How do you solve a quadratic equation if it *cannot* be factored?

Intermediate algebra
Class notes
Solving Quadratic Equations by the Square Root Method and Completing the Square (section 18.1)

Definition: Quadratic equation: A quadratic equation is an equation that could be written in the form $ax^2 + bx + c = 0$ where a is *not* zero.

What kind of equation does it become when a is zero?

expls: $5x^2 + 2x + 16 = 0$
$(x + 5)(x - 8) = 0$
$4x^2 + 48x = 12$
$-5x^2 + 4x - 21 = 0$

These last three are not in the form $ax^2 + bx + c = 0$. Are they really quadratic equations?

counterexamples: $\frac{x-3}{x+9} = \frac{6}{x+2}$, $\sqrt{4x+5} = 7$, $2|5x - 3| + 7 = 21$, $5x + 8 = 0$

When we encountered quadratic equations before, we were able to get zero on one side, factor the other side, and then break apart the equation into two simpler, usually linear, equations that we knew how to solve.

Most of the equations we see here will not be factorable, so we need other methods. We will investigate the square root method, completing the square, and the quadratic formula in the sections to come.

Square Root Method: Recall that to solve an equation like $\sqrt{4x+5} = 7$, you square both sides to undo the square root. We learned that "squaring" and "square rooting" undid each other. So it makes sense that when we want to solve equations like $x^2 = 25$ or $(x-9)^2 = 64$, we can square root both sides to get started. But there is one complication we must discuss.

Let's look at $x^2 = 25$ first.

Just looking at the equation, what do you know the solution is?

Did you guess **both** solutions?

Notice, both 5 and -5 make the equation true. We write this as $x = \pm 5$. Look at the solution below to see how the algebra works out.

$$x^2 = 25$$
$$\sqrt{x^2} = \sqrt{25}$$
$$?? = ??$$
$$x = \pm 5$$

Fill in the missing line.

What is $\sqrt{x^2}$ when *x* could be any real number?

What is $\sqrt{25}$?

The inclusion of the $\pm$ sign (pronounced "plus or minus") is very important. But in practice, we usually will not write the third line above. We will simply write

$$x^2 = 25$$
$$\sqrt{x^2} = \sqrt{25}$$
$$x = \pm 5$$

expl 1: Solve.

$$(x-9)^2 = 64$$

Square root both sides.

What is $\sqrt{64}$?

Don't forget the $\pm$ sign.

Finally, get *x* totally alone.

Did you get $\pm 8+9$? So, what does that mean? Interpret it as -8 + 9 and 8 + 9. So our two solutions are 1 and 17. Check them individually in the original equation. Do they work?

Check $x = 1$: $(x-9)^2 = 64$

Check $x = 17$: $(x-9)^2 = 64$

expl 2: Solve.

$x^2 - 48 = 0$

Factoring does not help here. So we'll use our new method.

Isolate the x^2 first.

Don't forget the $\pm$ sign.

What is $\sqrt{48}$?

Use the calculator to check your answers.

What do you hope this is equal to? Did $4\sqrt{3}$ work?

Calculator: $\left(4\sqrt{}(3)\right)^2 - 48$ [ENTER]

Now, press [2nd] [ENTER] to get your last entry back again. We will insert a negative sign before the 4 to change it to $-4\sqrt{3}$. Arrow over to the 4 and press [2nd] [DEL]. Notice the second function of DEL is INS, which stands for INSERT. Your cursor should change. Press the negative sign (in the number pad) and press [ENTER]. Your screen should look like the following.

$\left(-4\sqrt{}(3)\right)^2 - 48$

Did $-4\sqrt{3}$ work?

expl 3: Solve.

$(x+14)^2 = -144$

Think about the equation before plunging ahead. Will the solution be a real number?

Did you get $x = -14 \pm 12i$? The i in the solutions indicate they are complex (or imaginary) numbers. Does that make sense, considering the original equation? Why?

expl 4: Solve.

$(4x+9)^2 = 6$

Think about what happened to *x* to get it into the equation. Undo that in reverse order.

Don't forget the $\pm$ sign.

expl 5: Solve.

$3p^2 - 36 = 0$

Factoring will not help much here. Isolate the p^2 and then square root both sides.

Don't forget the $\pm$ sign.

Completing the square: In the previous examples, we were able to square root both sides to help isolate the variable. But that does not work with equations like $x^2 + 12x = -25$. How do we solve this?

> If only this was in the form (something)^2 = blah blah blah.

Completing the square is a technique that forces the left side of the equation into the form $(x + ?)^2$. From there, we square root both sides like we did before. Before we attack this problem, let's look at why completing the square works.

Look at this pattern: FOIL these problems.

$(x + 4)^2 =$

$(x + 7)^2 =$

$(x - 5)^2 =$

> Recall these are perfect square trinomials.

So, we are interested in going from $x^2 + 8x + 16$ back to the $(x + 4)^2$ form. But what if we were just given $x^2 + 8x$?

How would we figure out the constant that "completed" $x^2 + 8x$ so that we could factor it as $(x + 4)^2$?

In each trinomial above, what is the relationship between the coefficient of the x-term and the constant at the end?

What would you add to $x^2 + 12x$ so that we could write it as $(x + ?)^2$, and what goes in the parentheses?

Now that we have the general idea of completing the square, let's use it to solve some equations.

expl 6: Solve.

$x^2 + 12x = -25$

We want to turn the left side into $(x + ?)^2$. How?

If we add something on the left, we must do the same to the right.

Once you have the form $(x + ?)^2 = ?$, we solve like before.

expl 7: Solve.

$x^2 - 50 = 2x$

We need the x^2 and x-terms on the same side, and the constant on the other.

expl 8: What constant would be added to the following expression to "complete the square"? In other words, what constant should we add so that the resulting expression is a perfect square trinomial? Use a fraction; *not* a decimal answer. Add your constant on and then factor the expression.

$t^2 + 13t$

expl 9: Solve by completing the square.
$2p^2 - 6p + 1 = 0$

We must factor out the 2 from the first two terms before "completing the square".

expl 10: The formula for the amount of money in an account that earns annually compounded interest is $A = P(1+r)^t$.

Here, A is the amount in the account after t years, P is the amount initially invested, and r is the annual interest rate.

This is a simplified version of the formula where we assume the account is compounded annually (once a year).

a.) Find the rate you would need to earn in order to grow \$60,000 into \$100,000 in two years. Round to the nearest percent.

b.) If you needed *exactly* \$100,000 in two years, would your answer be sufficient? Explain. (Hint: Check your answer.)

The world is often messier than math books would have us believe.