We will multiply, divide, and simplify radicals. The distance formula uses a radical. The midpoint formula is just good fun.

Intermediate algebra
Class notes
Simplifying Radical Expressions and the Distance and Midpoint Formulas (section 17.3)

We will work with two important rules for radicals. We will write them for square roots but they work for any root (cube root, fourth root, etc.).

Product and Quotient Rules: For any real numbers a and b, we have $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$.

Likewise, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$. (This assumes that these radicals are real numbers, meaning that a and b are non-negative. In the case of the second rule, b must be non-zero. Why?)

This is just a matter of order of operations. Do you multiply before square rooting, or square root then multiply? The rule says it should not matter.

It's always a good idea to investigate rules instead of believing them blindly. Use the examples below to verify the rules for yourself.

expl 1: Putting 4 in for a and 25 in for b, the product rule becomes $\sqrt{4 \cdot 25} = \sqrt{4} \cdot \sqrt{25}$. Simplify each side separately to verify the rule.

Multiply, and then take the square root.

Square root each, and then multiply them.

Were they equal? Does the rule hold true?

expl 2: Putting 64 in for a and 16 in for b, the quotient rule becomes $\sqrt{\frac{64}{16}} = \frac{\sqrt{64}}{\sqrt{16}}$. Simplify each side to verify the rule.

Divide, and then take the square root.

Square root each, and then divide them.

Were they equal? Does the rule hold true?

expl 3: Use the product rule to multiply the following. Simplify if possible.

a.) $\sqrt{5} \cdot \sqrt{3x}$

Rewrite as one radical, and then simplify what you can.

b.) $\sqrt[3]{10} \cdot \sqrt[3]{3}$

Rule works for any index: $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$

c.) $\sqrt[4]{ab^2} \cdot \sqrt[4]{27ab}$

expl 4: Use the quotient rule to divide the following. Simplify if possible.

a.) $\dfrac{\sqrt{26}}{\sqrt{2}}$

b.) $\dfrac{\sqrt{a^7 b^6}}{\sqrt{a^3 b^2}}$

Simplify radicals using the new rules: We will factor the radicand to find perfect squares (or perfect *n*th powers as in examples 6 through 8 below). These can be removed from the radical. Leftover bits will stay under the radical.

Look for perfect squares, cubes, or *n*th powers.

expl 5: Simplify. Assume variables represent positive numbers.

$\sqrt{75x^8}$

$75 = 25 \cdot 3$

$x^8 = \left(x^4\right)^2$

$\sqrt{25} = 5$

$\sqrt{x^8} = x^4$

expl 6: Use the quotient rule to divide the following. Simplify if possible.

$$\frac{\sqrt[5]{64x^{10}y^3}}{\sqrt[5]{2x^3y^{-7}}}$$

What are the perfect 5th powers?

Rule works for any index:

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

expl 7: Simplify. Assume variables represent positive numbers.

$$\sqrt[4]{162x^4y^5}$$

Our new rules allow us to look at the factors individually.

$\sqrt[4]{y^5} = \sqrt[4]{y^4 \cdot y} = ??$

$\sqrt[4]{x^4} = ??$

Find a factor of 162 which is a perfect 4th power.

$2^4 = 16$

$3^4 = 81$

$4^4 = 256$

$5^4 = 625$

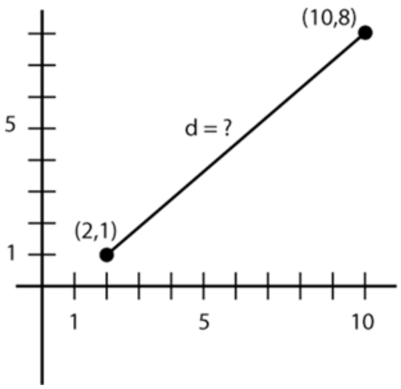
$\vdots$

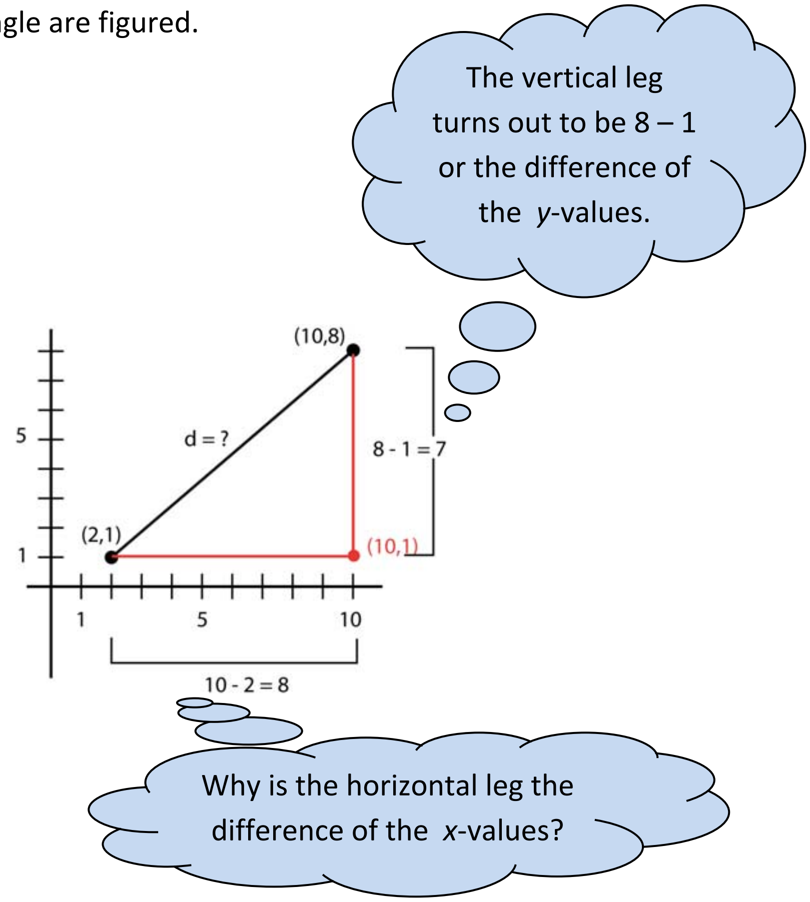
expl 8: Simplify. Assume variables represent positive numbers.

$$\sqrt[3]{\frac{2x}{-81y^{12}}}$$

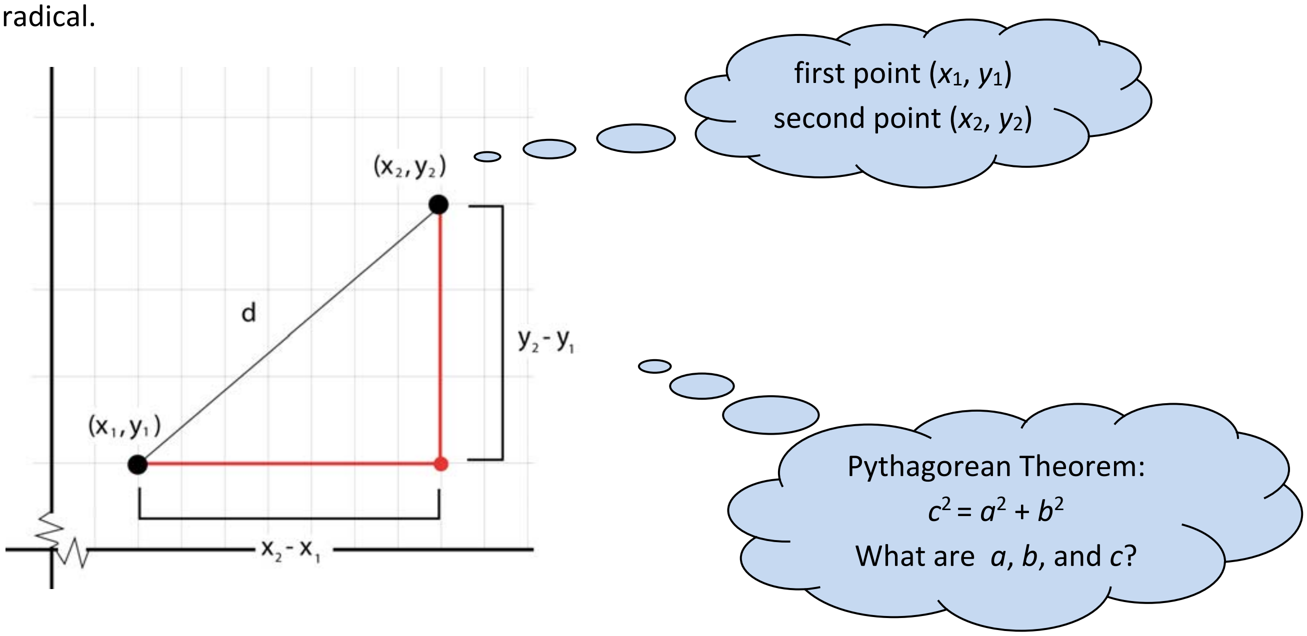
Find a factor of -81 which is a perfect 3rd power (cube). What do you do with the 2*x* on top?

Distance Formula: Below are two points on a plane. If we want to find the distance between them, we can draw in a right triangle (picture on right) and use the Pythagorean Theorem. Notice in particular how the legs of the triangle are figured.





We will now use two points in general to derive the formula for the distance between them. We will again use the Pythagorean Theorem. This is in this section because it will involve a radical.



Follow the steps below to derive the distance formula. We start with the Pythagorean Theorem and replace the variables with the pieces of our triangle.

$c^2 = a^2 + b^2$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Put d in for the hypotenuse, and the legs in for a and b.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Square root both sides to get d alone.

expl 9: Use the distance formula to find the distance between the two points (5, -4) and (10, 6). Give an exact distance and a three-decimal place approximation.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

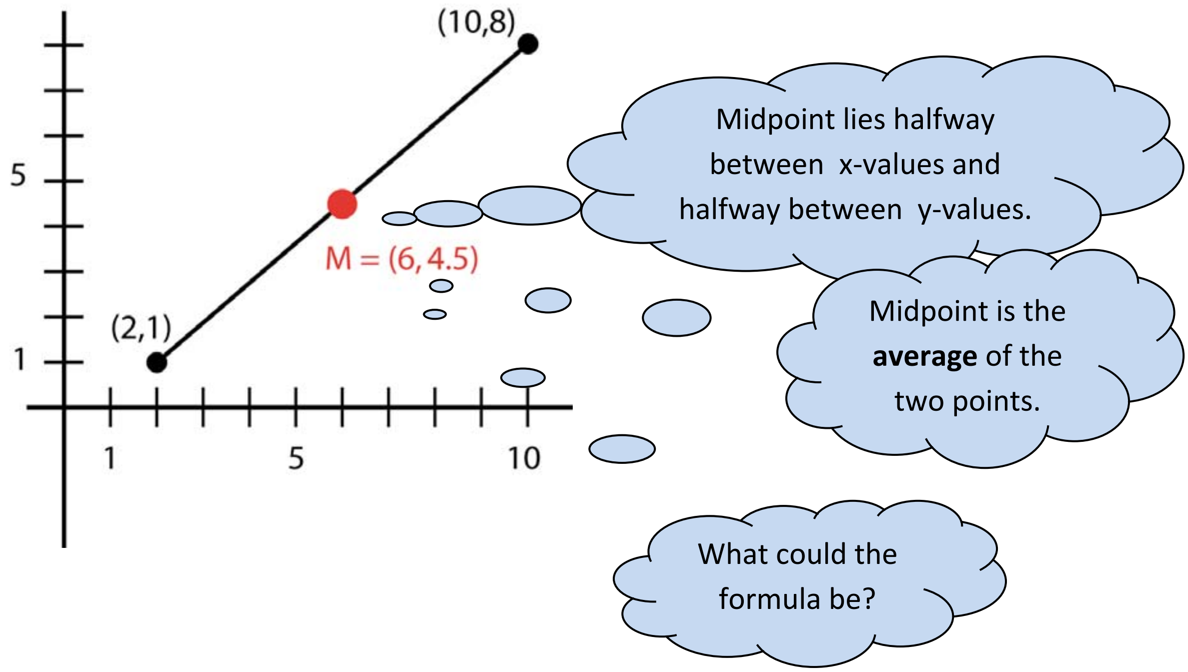
Your points are (x_1, y_1) and (x_2, y_2).

Mind the order of operations!

Reduce the radical to get the exact distance.

Use your calculator for the approximation.

Midpoint Formula: This gives the point in the exact middle of two other points. (It is said to be the midpoint of the segment connecting the two points.) See the example below. Can you come up with a formula for the points (x_1, y_1) and (x_2, y_2)?



expl 10: Find the midpoint of the line segment whose endpoints are (9, -5) and (4, 3). Also quickly plot the two points with your midpoint to check yourself.