What is the square root of a negative number?

Intermediate algebra
Class notes
Complex Numbers (section 17.7)

When we evaluate $\sqrt{25}$, we ask ourselves "What (non-negative) number squared makes 25?"

But what about $\sqrt{-25}$? What number squared makes -25 ?

$(?)^2 = -25$

It turns out that no real number can be squared to make -25. But instead of leaving problems with square roots of negative numbers unresolved, we made up complex or imaginary numbers. We start off by defining the basic building block of complex numbers.

We will let $i = \sqrt{-1}$.

So, now we can simplify $\sqrt{-25}$ as $\sqrt{25}\sqrt{-1}$ or $5i$.

By the way, what do you think i^2 is equal to? This fact will be useful to us when we manipulate complex numbers.

If $i = \sqrt{-1}$, then $i^2 = \sqrt{-1}\sqrt{-1}$.

If $\sqrt{9}\sqrt{9} = 9$, then what is $\sqrt{-1}\sqrt{-1}$?

How would you verify that $5i$ is the number that you square to get -25? Do it now.

expl 1: Write the following in terms of *i*.

a.) $\sqrt{-36}$

c.) $\sqrt{-32}$

b.) $-\sqrt{-36}$

d.) $3\sqrt{-75}$

expl 2: Multiply or divide.

a.) $\sqrt{-4}\sqrt{-25}$

You **cannot** assume the rules $\sqrt{a}\sqrt{b} = \sqrt{ab}$ and $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ work here. They only work for **non-negative** values of a and b. (And b must be non-zero as well for the quotient rule.) You'll have to go another route.

b.) $\frac{\sqrt{-64}}{\sqrt{4}}$

Definition: Complex number: A complex number is a number that could be written in the form $a + bi$ where a and b are real numbers.

expls: $0 + 5i$ $\quad$ $2 + 4i$ $\quad$ $-6 - 7i$ $\quad$ 14 $\quad$ $\sqrt{5} + 4i$

a : real part
bi : imaginary part

Can you see how each could be written in the form $a + bi$? What are a and b in each case?

Definition: Conjugate of a complex number: The conjugate of the complex number $a + bi$ is said to be $a - bi$.

Simply change the sign of the i term to form the conjugate.

expl 3: Write the conjugates of each complex number below.

$5i$ $\quad$ $2 + 4i$ $\quad$ $-6 - 7i$ $\quad$ 14 $\quad$ $\sqrt{5} + 4i$

What is the product of a complex number and its conjugate? Find out by FOILing out $(a + bi)(a - bi)$. Is the product a complex number or just a real number? How can you tell?

We will use this fact when we divide complex numbers. But first, let's learn to add, subtract, and multiply them.

expl 4: Add or subtract. Write the result in the form $a + bi$.

a.) $(2 + 4i) + (6 - 3i)$

What would you do if it was $(2 + 4x) + (6 - 3x)$?

b.) $(-5 + 7i) + (-9i)$

c.) $(2 + 4i) - (6 - 3i)$

expl 5: Multiply. Write the result in the form $a + bi$.

a.) $-5i \cdot 6i$

What would you do if it was $(4 + 3x)(5 - 4x)$?

b.) $(4 + 3i)(5 - 4i)$

What was i^2 again?

c.) $\left(\sqrt{5} + 4i\right)\left(\sqrt{5} - 4i\right)$

Dividing complex numbers involves eliminating the i from the bottom of the fraction. Recall that multiplying a complex number by its conjugate does just that. We will have to simplify a bit more to get the final answer in the form $a + bi$.

expl 6: Divide. Write the result in the form $a + bi$.

a.) $\dfrac{5}{2+3i}$

What is the conjugate of $2 + 3i$?

b.) $\dfrac{2+5i}{3+3i}$

What is the conjugate of $3 + 3i$?

Simplify, simplify, simplify…

c.) $\dfrac{5}{2i}$

What is the conjugate of $2i$? But we can use something simpler. What?

Powers of *i* : So $i^1 = i$ and $i^2 = -1$, but what are the higher powers of i ? Fill in the following powers of i to elicit a pattern.

Recall $i^3 = i^2 \cdot i$,
$i^4 = i^2 \cdot i^2$,
$i^5 = i^2 \cdot i^3$,
etc.

$i^1 = i$	$i^5 =$	$i^9 =$	$i^{13} =$
$i^2 = -1$	$i^6 =$	$i^{10} =$	$i^{14} =$
$i^3 =$	$i^7 =$	$i^{11} =$	$i^{15} =$
$i^4 =$	$i^8 =$	$i^{12} =$	$i^{16} =$

expl 7: Use the pattern above to find each of the following powers of *i*.

a.) i^{21}

Where would i^{21} fall in the table above?

b.) i^{84}

Continue the pattern above to the left side for negative (and zero) powers.

c.) i^{-6}

d.) $(3i)^7$

Need your exponent rules, perhaps?

Optional Worksheets: Manipulating complex numbers and Manipulating complex numbers 2

These worksheets (solutions available online) will help practice finding powers of i and operating with complex numbers. They also investigate one application of complex numbers, the complex solutions to quadratic equations.

Worksheet: Manipulating complex numbers 3

We will practice adding, subtracting, multiplying, and dividing complex numbers. We will also explore the powers of i.