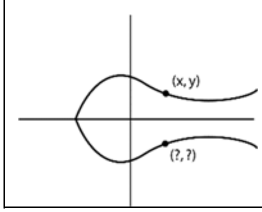
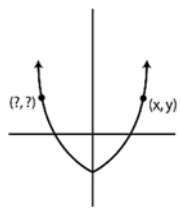
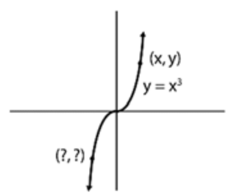
We will talk about symmetry, when a function is going up or down, extreme values, and how fast a function is changing.

College Algebra
Class Notes
Properties of Functions (section 3.3)

We will revisit the concept of symmetry and explore concepts such as the largest or smallest y-value on a graph. The concepts are intuitive but we will use algebra to describe them.

Worksheet: Investigating functions 3:
We work on the definition of a function, domain, and finding function values graphically and algebraically.

Recall: Symmetry: Do you remember which graph below is symmetric about the y-axis? Which is symmetric about the x-axis? Which is symmetric about the origin?

			Do you remember the coordinates of the unknown points?

Recall: Algebraic tests for symmetry:
The pictures above help justify the following tests.

To test a relationship for symmetry about the …

***x*-axis:** Replace y with $-y$. If the equation is equivalent, then the relationship is symmetric with respect to the x-axis.

***y*-axis:** Replace x with $-x$. If the equation is equivalent, then the relationship is symmetric with respect to the y-axis.

origin: Replace x with $-x$ and replace y with $-y$. If the equation is equivalent, then the relationship is symmetric with respect to the origin.

We are focusing on functions. Which above graph and symmetry could *not* be of a function? Cross it off because we will think of it no more. No more, I say!

We have the following definitions.

Definition: Even and Odd Functions:

Picture
$y = x^2$ and $y = x^3$.
Draw them here.

If a function is symmetric about the y-axis, we call it **even**.

If a function is symmetric about the origin, we call it **odd**.

We can frame the earlier algebraic test in terms of function notation.

A function is **even**, if and only if for every x in the domain, we know $f(-x) = f(x)$.

A function is **odd**, if and only if for every x in the domain, we know $f(-x) = -f(x)$.

What does "if and only if" mean?

We'll use this to check whether a function is even, odd, or neither.

Could a function be both even and odd?

expl 1: For the following functions, test if it is even, odd, or neither.

a.) $f(x) = x^3 + x$

Find $f(-x)$. Is it equal to $f(x)$ or $-f(x)$ or neither?

b.) $f(x) = 3x^2 - 5x^4$

c.) $f(x) = x^2 + 3x - 4$

Here's an optional problem to stretch your mind.

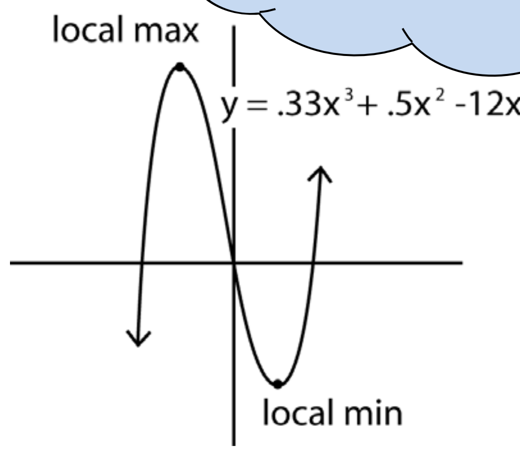
expl 2: Prove that the product of an odd function and an even function will always be odd.

Let $f(x)$ be an even function and $g(x)$ be an odd function.

We need to show
$$(f \cdot g)(-x) = -(f \cdot g)(x)$$
OR
$$f(-x) \cdot g(-x) = -f(x) \cdot g(x)$$

Definition: Local (or relative) extrema:
A **local minimum** is the point (technically, the y-value) on the graph where the ***y*-value is the smallest**, in that area of the graph.

A **local maximum** is the point (technically, the y-value) on the graph where the ***y*-value is the largest**, in that area of the graph.



expl 3: Use your calculator to find the local maximum and minimum of the function pictured above. Do *not* just TRACE but rather use the Maximum and Minimum calculator functions.

Use the Minimum (or Maximum) function (under the CALC menu on the TI-83 or 84) to find the smallest (or largest) y-value and the x-value that makes it.

Worksheet: Finding maximums and minimums on your graphing calculator (82, 83, 85, 86):

This worksheet shows how to find the points of maximum or minimum y-values on your graphing calculator. Instructions for the TI83 will work for TI84's.

Local (Relative) Extrema: A more precise way to state this definition:

Suppose f is a function for which $f(c)$ exists for some c in the domain of f. Then

> $f(c)$ is a **local (or relative) minimum** if there exists an open interval I containing c such that $f(c) \leq f(x)$ for all x in I; or
> $f(c)$ is a **local (or relative) maximum** if there exists an open interval I containing c such that $f(c) \geq f(x)$ for all x in I.

Absolute Extrema (Minimums and Maximums):

The concept of local maxes and mins focuses on a small interval around a given point. Consider the definition here and how it differs.

Definition: Absolute Maximums and Minimums:

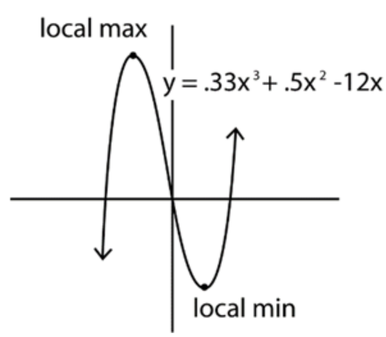
Let f be some function defined on the interval I (meaning I is the whole domain of f).

> If there is a number u in I for which $f(u) \geq f(x)$ for all x in I, then f has an absolute maximum at u, and the number $f(u)$ is the **absolute maximum** of f on I.
> If there is a number v in I for which $f(v) \leq f(x)$ for all x in I, then f has an absolute minimum at v, and the number $f(v)$ is the **absolute minimum** of f on I.

This max or min has to be the largest or smallest on the *whole* graph, *not* just an interval surrounding it.

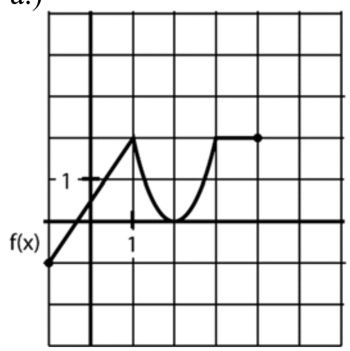
Again, it is the y-value that is said to be the max or min.

Think back to this graph. What are the absolute maximum and minimum? Do they even exist?

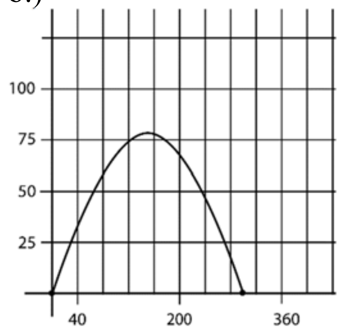


expl 4: For each function below, determine the absolute maximum and absolute minimum if they exist. If an extremum does *not* exist, explain why.

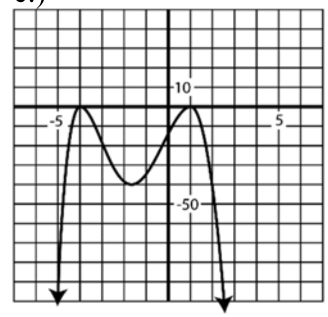
a.)



b.)



c.)

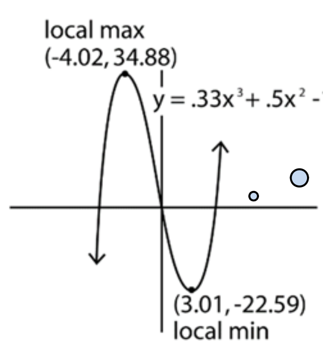


By the way, for part *b*, the endpoints' *y*-values of 0 are *not* considered to be a *local* min because there is *not* an *open* interval containing it. This does *not* come up when finding the absolute minimum.

Increasing and decreasing functions:
We will investigate where the graph's *y*-values are increasing and where they are decreasing.

We look at **what is happening to the *y*-values as we go left to right** on the graph. And then we **write the intervals of *x*-values** that result in those increasing or decreasing parts of the graph. We will usually use interval notation.

expl 5: Where is this function increasing and decreasing? Write your answers in interval notation.



Think left to right.

A more precise way to state this definition:
A function is **increasing** on an interval I, if for any choice of x_1 and x_2 in that interval where $x_1 < x_2$, then $f(x_1) < f(x_2)$.

A function is **decreasing** on an interval I, if for any choice of x_1 and x_2 in that interval where $x_1 < x_2$, then $f(x_1) > f(x_2)$.

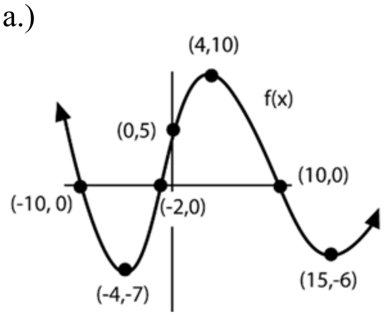
A function is **constant** on an interval I, if for all values of x in I, then the values of $f(x)$ are equal.

Some books will define these intervals to be strictly open. This book does *not*.

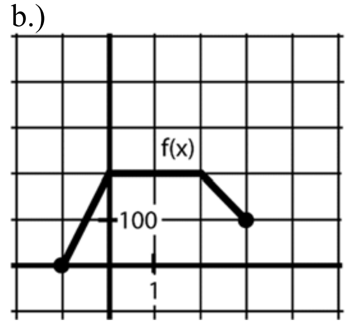
Think left to right.

expl 6: For each function below, determine the intervals where the function is increasing, decreasing, or constant. Write your answers in interval notation using square brackets.

a.)



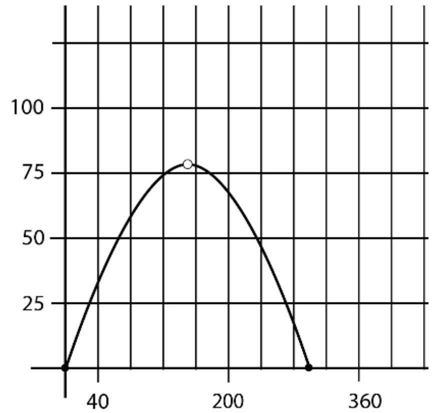
b.)



Remember these intervals are *x-values*.

Holes in graphs:

expl 7: Consider this amended graph from a previous problem. Notice the hole at the top of the parabola.



Here, the (approximated) point at the top (150, 80) is *no* longer considered a maximum. Do you see why?

Write the intervals where this graph is increasing or decreasing.

We should *not* include 150 so use half-open intervals.

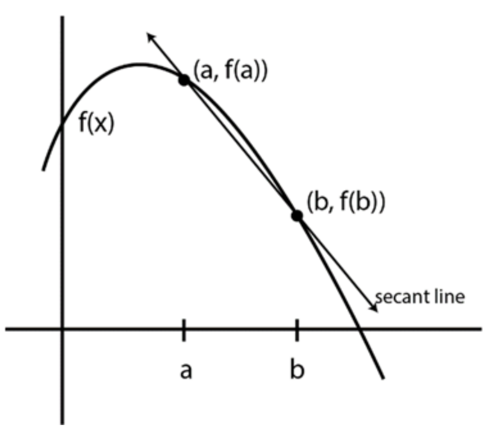
Average Rate of change:

You might recall that the slope of a straight line is the difference of the y-values divided by the difference of the x-values. This ratio tells us how fast y is changing with respect to x, or the **average rate of change**.

Calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

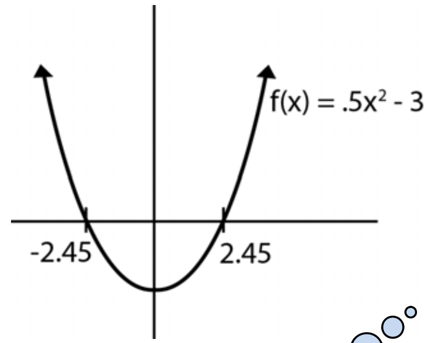
What if we used this to investigate a non-linear function? We can select two points on the graph and find the slope of the line between them (called the **secant line**). We will amend our formula for slope a bit, using function notation.



Definition: Average rate of change:

If a and b where $a \neq b$ are in the domain of $f(x)$, then the **average rate of change** of f from a to b is $\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}, \quad a \neq b$.

expl 8a: For the function pictured to the right, find the average rate of change from 1 to 4.



$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

b.) Do your best to plot the points we are talking about above. Draw the secant line whose slope you have just found.

Do you remember what it means graphically for a slope to be positive or negative?

expl 9: The US government's debt is a function of time that is increasing. In 2012, it was $16,066 billion. In 2018, it had grown to $21,516 billion. Find the average rate of change for these years. Describe, in words, what this average tells us about how much the debt grows each year.

Can you write $16,066 billion as a plain number?