College algebra
Class notes
Logarithmic Functions and Their Graphs (section 5.3)

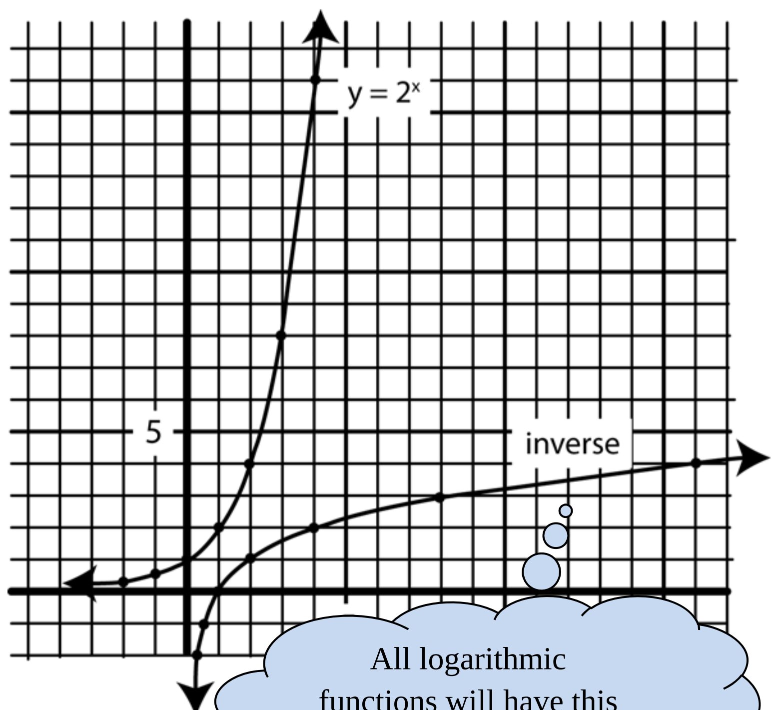
*We'll use this new name for the inverse of the exponential function.*

Let's investigate the inverse of the exponential function from the previous section. I have recreated the table of values and graphed the exponential function $y = 2^x$ below.

Then I switched the $x$ and $y$ values in the equation and table. I graphed the resulting inverse relation using the points from the table. Is this inverse a function?

| **Exponential** | | **Inverse** | |
|---|---|---|---|
| $x$ | $y = 2^x$ | $x$ | $y$ in $x = 2^y$ |
| -2 | ¼ | ¼ | -2 |
| -1 | ½ | ½ | -1 |
| 0 | 1 | 1 | 0 |
| 1 | 2 | 2 | 1 |
| 2 | 4 | 4 | 2 |
| 3 | 8 | 8 | 3 |
| 4 | 16 | 16 | 4 |

*Is the exponential function one-to-one? If so, its inverse will be a function.*




*All logarithmic functions will have this general shape.*

So, to find the equation for the inverse, we'd normally take the equation $x = 2^y$ and solve for $y$. But we do not know how to isolate $y$. So we'll invent new notation and write $y = \log_2 x$ to mean the same as $x = 2^y$. We need to be able to interpret this new log notation.

In words, how would you describe $y$ in the equation $x = 2^y$? Use the right-side table if you need. In other words, how is $y$ related to 2 and $x$?

We will use this idea to define what $\log_2 x$ means. We will say that $\log_2 x$ is the number to which I raise 2 to get $x$. This is very important in our study of logs.

expl 1: Use the fact that "$\log_a x$ is the number to which I raise $a$ to get $x$" to figure the following logs without the calculator.

"$\log_3 27$ is the number to which I raise ____ to get ____"

a.) $\log_3 27$

b.) $\log_5\left(\frac{1}{125}\right)$

c.) $\log_{10} \sqrt{10}$

d.) $\log_6 6^3$

This last one will lead us to a log rule in the next section.

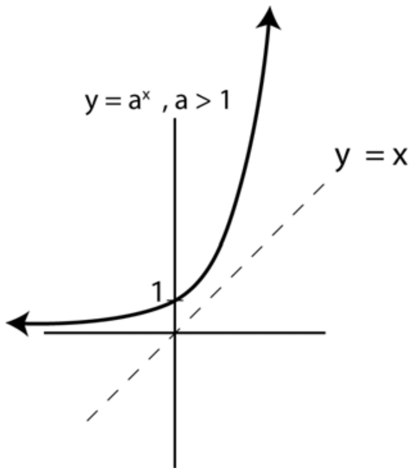
**Definition: Logarithmic Function:**
We define $y = \log_a x$ to be the number $y$ such that $x = a^y$. Because of its connection to the exponential relationship, we say $x > 0$ (this is the domain of the function) and $a$ is a positive constant not equal to 1.

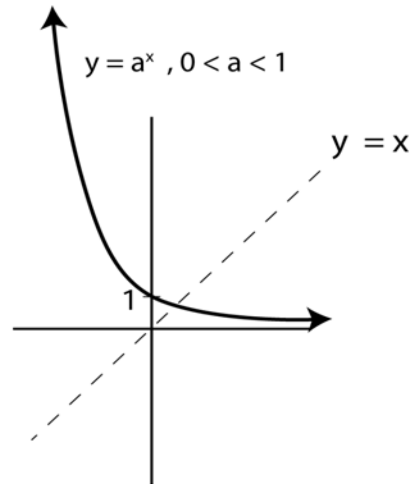
A more useful way to define logs, as stated above, is $\log_a x$ is the number to which I raise $a$ to get $x$.

The number $a$ is called the **base**. Notice it is the same as the base of the exponential function from which it came.

Sometimes written as $y = \log_a(x)$.

**Graphs:** The graphs of logarithmic functions will come in two different flavors, just like exponential graphs. Below are the graphs of the basic exponential functions. Reflect them over the line $y = x$ to get their logarithmic inverses.




Are these logarithmic functions one-to-one? What are their domains? What are their ranges? What are their $x$ and $y$-intercepts? Are they increasing or decreasing?

**Definition: Common Logarithmic Function:**
If $10$ is the base of the logarithm, we have $y = \log_{10} x$. We will call this the common logarithmic function. We abbreviate "$\log_{10}$" as simply "log" with no base apparent.

**Definition: Natural Logarithmic Function:**
If $e$ is the base of the logarithm, we have $y = \log_e x$. We will call this the natural logarithmic function. We abbreviate "$\log_e$" as "ln".

**Calculator usage:**
You will see two buttons on your calculator, LN and LOG. These are base $e$ and base 10 logs. To find logs of other bases, we will probably need a change-of-base formula discussed later.

expl 2: Find each using the calculator.
a.) log 650

b.) ln 80.56

c.) $\log_{10} 300$

d.) $\log_{10} -20$

Why does $\log_{10} -20$ give you an error?

**Change of Base Formula:**
You cannot find logs other than base 10 or $e$ on your calculator. We need another way. For any logarithmic bases $a$ and $b$, and any positive number $M$, we know that $\log_b M = \dfrac{\log_a M}{\log_a b}$.

Some newer calculators will calculate logs of any base. Look under the MATH menu for "logBASE".

[We will usually use 10 or $e$ for the base $a$ so we can do these problems on the calculator.]

expl 3: Use the change of base formula to find the following.
a.) $\log_3 12$

You can use log base 10 or $e$. Try both to see!

b.) $\log_5 100$

**Convert between exponential equations and logarithmic equations:**
We have the general notion that $y = \log_a x$ and $x = a^y$ are equivalent. That means, if we have an equation in exponential form, we should be able to convert it to logarithmic form using these equations as a guide, and vice versa.

You can also do this conversion by thinking about how "$\log_a x$ is the number to which I raise $a$ to get $x$".

expl 4: Convert the logarithmic equation to the equivalent exponential equation.

a.) $\log_{10} 10{,}000 = 4$

Keep in mind
$y = \log_a x \longleftrightarrow x = a^y$

b.) $\log_3 x = 4$

c.) $\log_x b = .845$

expl 5: Convert the exponential equation to the equivalent logarithmic equation.

a.) $4^x = 64$

This will help
us solve
equations later.

b.) $10^b = 9764$

c.) $4^6 = 4096$

expl 6: A model for advertising response is given by $N(a) = 1000 + 200 \ln a, \quad a \geq 1$. Here $N(a)$ is the number of units sold when $a$ thousand dollars is spent on advertising.

a.) How many units would be sold if they spend $5,000 on advertising?

What is $a$?

b.) Graph the function on the window [0, 25] x [0, 2000]. What happens to the number of units sold as $a$ increases?