We will see many ways to solve these familiar equations.

College algebra
Class notes
Solving Quadratic Equations: Factoring, Square Root Method, Completing the Square, and the Quadratic Formula (section 3.2)

In this section, we will solve equations. What does it mean to solve an equation? In other words, what are you doing when asked to solve the equation $x^2 - 11x + 24 = 0$?

We will investigate solving by factoring, the square root method, completing the square, and the quadratic formula. You will find that the equation's form will help determine the method that is best suited to solve it.

Definition: Quadratic equation: A quadratic equation is an equation that could be written in the form $ax^2 + bx + c = 0$ where a is not zero.

examples:
- $5x^2 + 2x + 16 = 0$
- $(x + 5)(x - 8) = 0$
- $4x^2 + 48x = 12$
- $-5x^2 + 4x - 21 = 0$

What kind of equation does it become when a is zero?

They may not *look* to be in the form $ax^2 + bx + c = 0$. Are they really quadratic equations?

counterexamples: $\dfrac{x-3}{x+9} = \dfrac{6}{x+2}$, $\sqrt{4x+5} = 7$, $2|5x - 3| + 7 = 21$, $5x + 8 = 0$

Solving Equations by Factoring:

Zero Factor Theorem: If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

This is the crux of the whole method. So make sure you believe it before going on. Put numbers in for a and b to make sense of it.

expl 1: Solve.

$(x + 2)(3x + 1) = 0$

See this as two numbers whose product is 0. The two numbers are "$x + 2$" and "$3x + 1$".

What do you know about "$x + 2$" and "$3x + 1$"? Use that to write two simpler equations. Then solve them to get x alone.

expl 2: Solve.

$2x^2 + 6 = -13x$

We want to use the Zero Factor Theorem. What needs to be done before we factor?

expl 3: Solve.

$3y^3 + 11y^2 - 4y = 0$

Factor, factor, factor…

Did you find all 3 solutions? Check them using your calculator.

Optional Worksheets: Here are two popular methods for factoring trinomials. A third worksheet shows a handy calculator shortcut for the first method, the AC method. Other less-known methods are also available on www.stlmath.com.

Factoring trinomials: Part 1: Introduction, Factoring by grouping, and A-C method
The AC method is sometimes referred to as factoring by grouping because the method uses it. However, my worksheets make a distinction between factoring by grouping and the AC method.

Factoring trinomials: Part 2: Reverse FOIL method
This method may be called trial and error. It is more complicated when the leading coefficient is not 1 since there are more possibilities. Keep organized and you will be fine.

Factoring trinomials: Using the Calculator (TI-82, 83, and 84) to Help with the AC Method
This will help find the numbers needed for the AC method without too much work.

Solving Equations by the Square Root Property:

When solving equations, we add to undo subtraction (when we solve $x-9=13$), we divide to undo multiplication (when we solve $3x=15$).
What undoes the "square" in equations like $x^2=25$ or $(x-9)^2=64$?

There is one complication we must discuss.

Let's look at $x^2=25$ first.

Just looking at the equation, what do you know the solution is?

Did you guess **both** solutions?

Notice, both 5 and -5 make the equation true. We write this as $x=\pm 5$. Look at the solution below to see how the algebra works out.

$$x^2 = 25$$
$$\sqrt{x^2} = \sqrt{25}$$
$$?? = ??$$
$$x = \pm 5$$

Fill in the missing line.

What is $\sqrt{x^2}$ when x could be any real number?

What is $\sqrt{25}$?

The inclusion of the $\pm$ sign (pronounced "plus or minus") is very important. I would like you to know why it happens, but in practice, we usually will not write the third line above. We will simply write

$$x^2 = 25$$
$$\sqrt{x^2} = \sqrt{25}$$
$$x = \pm 5$$

expl 4: Solve to find the exact solution(s).

$$x^2 + 16 = 0$$

Factoring does not help here. So we'll use our new method.

Isolate the x^2 first. Then square root both sides.

Don't forget the $\pm$ sign.

Use the calculator to check your answers.

Calculator: $(4i)^2 + 16$ $\boxed{\text{ENTER}}$

Did $4i$ work?

Now, press $\boxed{2^{\text{nd}}}$ $\boxed{\text{ENTER}}$ to get your last entry back again. We will insert a negative sign before the 4 to change it to $-4i$. Arrow over to the 4 and press $\boxed{2^{\text{nd}}}\boxed{\text{DEL}}$. Notice the second function of DEL is INS, which stands for INSERT. Your cursor should change. Press the negative sign (in the number pad) and press $\boxed{\text{ENTER}}$. Your screen should look like $(-4i)^2 + 16$

Did $-4i$ work?

expl 5: Solve.

$$(x-9)^2 = 64$$

Square root both sides.

Don't forget the $\pm$ sign.

Solving Equations by Completing the Square: In the previous examples, we were able to square root both sides to help isolate the variable. But that does not work with equations like $x^2 + 12x = -25$. How do we solve this?

If only this was in the form (something)^2 = blah blah blah.

Completing the square is a technique that forces the left side of the equation into the form $(x + ?)^2$. From there, we square root both sides like we did before. Before we attack this problem, let's look at why completing the square works.

Look at this pattern: FOIL these problems.

Recall these are perfect square trinomials.

$(x + 4)^2 =$

$(x + 7)^2 =$

$(x - 5)^2 =$

So, we are interested in going from $x^2 + 8x + 16$ back to the $(x + 4)^2$ form. But what if we were just given $x^2 + 8x$? How would we figure out the constant that "completed" $x^2 + 8x$ so that we could factor it as $(x + 4)^2$?

In each trinomial above, what is the relationship between the coefficient of the x-term and the constant at the end?

What would you add to $x^2 + 12x$ so that we could write it as $(x + ?)^2$, and what goes in the parentheses?

Now that we have the general idea of completing the square, let's use it to solve an equation.

expl 6a: Solve by completing the square.

$$x^2 + 12x = -25$$

We want to turn the left side into $(x + ?)^2$. How?

If we add something on the left, we must do the same to the right.

expl 6b: Solve by completing the square.

$3p^2 - 12 = 5p$

We need the p^2 and p-terms on the same side, and the constant on the other. Then complete the square.

Completing the square only works if the coefficient of p^2 is 1. How would you fix that?

Don't forget the $\pm$ sign.

The clean-up can be tricky. Mind your order of operations and fraction arithmetic.

Solving Equations by the Quadratic Formula:

Some old guy long ago (actually many old guys independently throughout time), solved the general quadratic equation $ax^2 + bx + c = 0$ (where a is not zero) and found the solutions to be $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. We will piggyback on their work to solve our own equations.

expl 7: Solve to find the exact solution(s).

$x^2 + 10x + 13 = 0$

What are a, b, and c? ($ax^2 + bx + c = 0$)

Mind your order of operations!

Leave your answer in reduced radical form.

expl 8: Solve to find the exact solution(s).

$x^2 + x + 7 = 0$

What are a, b, and c?

What happens under the radical? Are the solutions real?

Definition: Discriminant: In the quadratic formula, the expression under the radical, $b^2 - 4ac$, is called the discriminant. It can be used to glean information about an equation's solutions without fully solving it.

The question is, "How many real solutions does an equation have if the discriminant is positive?" In which example(s) did we see that happen?

What about when the discriminant is negative? How many real solutions would those equations have? What was the nature of the solutions? In which example(s) did we see that happen?

What about when the discriminant is zero? What happens to the solutions $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ when $b^2 - 4ac$ is 0? Are the solution(s) real or imaginary (complex but not real)?

expl 9: Use the discriminant to determine the number and type of solutions to the equation below. In other words, determine if there is one real solution, two different real solutions, or two different imaginary solutions.

$3 - 6x^2 = -5x + 1$

What are a, b, and c?
($ax^2 + bx + c = 0$)
Be careful!

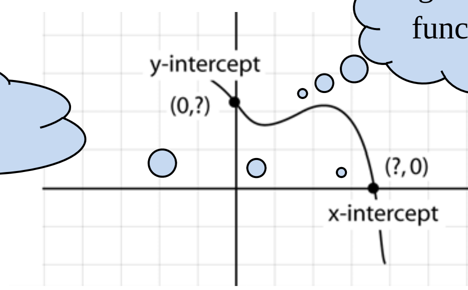
Only calculate $b^2 - 4ac$.
Do not use the whole quadratic formula.

Finding x-intercepts and zeros:

How do we find the x-intercept(s) of a function like $y = 2x^2 + 13x + 6$? The picture below may help you remember.

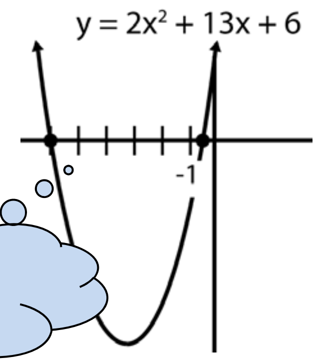
generic function

How would you find x when you know y must be 0?



expl 10: Find the x-intercepts of $y = 2x^2 + 13x + 6$.

Algebraically, put 0 in for y and solve.



Graphically, look at the x-axis.

Distinction between x-intercepts and zeros: The x-intercepts of a graph are written in ordered pair notation as points. The zeros of the function are said to be the x-values of these points.

Only real zeros will occur on the graph as x-intercepts. Complex zeros will not.

expl 11: Let $g(x) = x^2 + 10x + 13$. Use the quadratic formula to find the x-intercepts (and zeros) without graphing. Give exact answers and approximations. Check yourself on your calculator.

We solved the necessary equation in example 7.

Calculator Program: QUADRATC

This program will calculate the real and/or complex solutions to a quadratic equation. After you practice the quadratic formula by hand several times, feel free to use the calculator to solve equations. I will link to your calculator and give you the program. It is available online (called "TI83 [or TI82 or TI84] Quadratic formula program QUADRATC") and you could program it yourself but it is quite long.

Optional Worksheet: Link instructions for the TI calculators

This has general instructions on how to link two calculators and copy programs from one to the other.

Applications: Often a story problem will result in a quadratic equation. If this equation proves hard to factor or solve by previous methods, you'll want to use the quadratic formula.

expl 12: An isosceles right triangle has a hypotenuse that is 2 inches longer than either leg. Find the dimensions of the triangle.

The legs of an isosceles right triangle are equal in length. Draw a picture.

Old Pythagoras would have known where to start!

Solving Equations by using Quadratic Methods:

We will be solving equations that, once we manipulate them a bit, look quadratic. We can then use the methods we learned in this section to solve them.

***m*-Substitution:** This technique will use a dummy variable to take the place of a complicated expression that repeats within the equation we want to solve.

expl 13: Solve.

$$x^{2/3} - 5x^{1/3} + 6 = 0$$

What would you do if it was $m^2 - 5m + 6 = 0$?

If you substitute *m* for $x^{1/3}$, you'll see the equation becomes $m^2 - 5m + 6 = 0$.

You're not done when you find *m*. You still need to find *x*.

Solving equations graphically: When we solve an equation, we want to know the *x*-value(s) that makes the left side equal the right side, right?

So the simplest way to solve an equation graphically is to graph y = "left side of equation" and y = "right side of equation" and see where they intersect. Draw a quick, labeled graph and plot and label the solutions.

expl 14: Solve graphically. Round solutions to three decimal places.
$3p^2 - 12 = 5p$

Always use x as the variable on the calculator.

Do you know how to convert a graphical solution to a fraction?

Optional Worksheet: Roots and Intersections on your Calculator (82, 83, 85, 86):
Here we explore finding roots (commonly called zeroes or *x*-intercepts) of a single function and the intersection of two functions. We use these skills when we solve equations graphically. Instructions for the 83 will work for the 84 too.