What does it mean for $p \to q$ to be true? Can we complete a truth table for $p \to q$? What about $p \leftrightarrow q$?

General Education Mathematics
Class Notes
Logic: The Conditional and Biconditional (Section 3.3)

We have looked at truth tables for the first three connectives. Here, we consider the conditional and biconditional. When they are true might surprise you.

Recall: Definitions: A **conditional** expresses the notion of *"if . . . then"*. We use the arrow $\to$ to represent a conditional. A **biconditional** represents the idea of *"if and only if"*. We use a double arrow $\leftrightarrow$ to denote this symbolically. This double arrow implies that the logic flow goes both ways. We can think of $p \leftrightarrow q$ as *both* "if p, then q" and *also* "if q, then p".

Pronounce $p \to q$ as "if p, then q".

Pronounce $p \leftrightarrow q$ as "p if and only if q".

Definition: For a conditional $p \to q$, the statement p is called the **hypothesis** and q is called the **conclusion**.

We will see that a conditional is false *only* when the hypothesis is true and the conclusion is false.

Let's start off with a typical example of an "if…then" statement.

Mr. Gates, the owner of a small factory, has a rush order that must be filled by next Monday and he approaches you with this generous offer. He says, "If you work for me on Saturday, then I'll give you a $100 bonus."

If we let w represent "You work for me on Saturday" and b represent "I'll give you a $100 bonus," then this statement has the form $w \to b$.

Truth Table for Conditionals:

So, we let w represent "You work for me on Saturday" and b represent "I'll give you a \$100 bonus," and we are investigating the form $w \to b$.

Let's complete a truth table.

w	b	$w \to b$
T	T	
T	F	
F	T	
F	F	

We must examine four cases to determine exactly when Mr. Gates is telling the truth and when he is *not*.

Case 1 (*w* is true and *b* is true.):
You come to work and you receive the bonus. In this case, Mr. Gates certainly made a truthful statement. Fill in the table (row 1) with a T.

Case 2 (*w* is true and *b* is false.):
You come to work and you *don't* receive the bonus. Mr. Gates has gone back on his promise, so he has made a false statement. Fill in the table (row 2) with a F.

Case 3 (*w* is false and *b* is true.):
You *don't* come to work, but Mr. Gates gives you the bonus anyway. This is row 3 of the table.

Think carefully! Do *not* read more into his statement than he actually said. You do *not* expect to get the bonus if you did *not* come to work because that is your experience in everyday life. However, Mr. Gates *never said that*. *You would be assuming this condition.*

Remember that in logic, a statement is either true or false. Mr. Gates did *not* say something false; therefore, he has told the truth. Fill in the table (row 3) with a T.

Case 4 (*w* is false and *b* is false.):
You *don't* come to work and you *don't* receive the bonus. In this case, Mr. Gates is telling the truth for exactly the same reason as in Case 3, because he has *not* told a falsehood. Fill in the table (row 4) with a T.

If you do *not* come to work, Mr. Gates can give you the bonus *or not* give you the bonus. In either case, he has *not* told a falsehood and therefore is telling the truth.

Again, a conditional is false *only* when the hypothesis is true and the conclusion is false.

expl 1: Construct a truth table for the statement $(p \vee r) \rightarrow (p \wedge \sim q)$. Some of the table is filled in but you must determine some column headings.

p	**q**	**r**				**$(p \vee r) \rightarrow (p \wedge \sim q)$**
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

Step by step,
build the pieces
of the statement.

expl 2: Without bothering with a whole truth table, determine the truth value for $(\sim p \wedge q) \rightarrow r$ if the following conditions are met.

Conditions: p is true, q is false, and r is true

The negation symbol affects as
little of the expression as
possible. For example, $\sim p \wedge q$
would mean $\sim(p) \wedge q$
and *not* $\sim(p \wedge q)$.

Alternative Wording of Conditionals:

There are many ways to phrase a conditional without using the words “if…then”. Each of these forms is equivalent to “if p, then q”. Recall, the statement p is called the hypothesis and q is called the conclusion.

Alternative Phrasings for “if p, then q”	
q if p	The “if” is still associated with p even though it occurs later.
p only if q	“Only if” is *not* the same as “if”. The “if” condition is the hypothesis. The “only if” condition is the conclusion.
p is sufficient for q	The sufficient condition is the hypothesis.
q is necessary for p	The necessary condition is the conclusion.

Assume “If it rains, then I drive you home” is true.
These alternatives are “I drive you home if it rains.”
“It rains only if I drive you home.”
“It raining is sufficient for me to drive you home.”
“Me driving you home is necessary for it to rain.”

From Wikipedia:
When we write $p \to q$, we say that p is sufficient for q. This means that p being true is enough (sufficient) to assume that q is also true. We also say that q is necessary for p. This means that if q is *not* true, then p *cannot* be true either.

This last sentence is the contrapositive.

Quoting Wikipedia, and adding emphasis of my own,

> In ordinary English, "necessary" and "sufficient" indicate relations between conditions or states of affairs, *not* statements. For example, being a *male* is a necessary condition for being a brother, but it is *not* sufficient -- while being a *male sibling* is a necessary *and* sufficient condition for being a brother.

expl 3: Rewrite each statement using the words “if…then”.
a.) I’ll take a break *if* I finish my workout.

Finishing the workout is the *only* condition for taking a break.

b.) I’ll take a break *only if* I finish my workout.

Truth Table for Biconditionals:

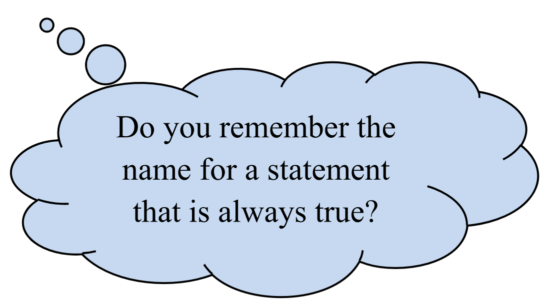
The statement "I will chop wood *if and only if* you will build a fire" says, "If I chop wood, then you will build a fire" *and* "If you will build a fire, then I will chop wood." We are saying, essentially, these two things are one and the same. If one happens, the other will too.

The biconditional $p \leftrightarrow q$ is true if both p and q have the same truth values (both true *or* both false). Otherwise, it's false. Let's fill in the truth table.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \leftrightarrow q}$
T	T	
T	F	
F	T	
F	F	

expl 4: Construct a truth table for the statement $(p \rightarrow \sim q) \leftrightarrow (q \rightarrow \sim p)$. Some of the table is filled in but you must determine some column headings.

$\boldsymbol{p}$	$\boldsymbol{q}$					$\boldsymbol{(p \rightarrow \sim q) \leftrightarrow (q \rightarrow \sim p)}$
T	T					
T	F					
F	T					
F	F					



expl 5: So, can we work backwards? Assume that p is false and $\sim p \rightarrow (p \vee q)$ is true. Deduce the truth value for q.

p	q	$\sim p$	$p \vee q$	$\sim p \rightarrow (p \vee q)$
T	T			
T	F			
F	T			
F	F			

What would the truth table for $\sim p \rightarrow (p \vee q)$ look like?

Only certain rows of this table apply to the situation we have. Which are they? What is the truth value for q?

Derived Forms of the Conditional:
1. The Converse:

The conditional $p \rightarrow q$ leads the mind to wandering and we start to think about the related statement $q \rightarrow p$.

Let's consider the conditional "If it rains, then I will drive you home." Define p and q and write, in words, the statement $q \rightarrow p$.

Definition: Consider the conditional $p \rightarrow q$. The **converse** of the statement is $q \rightarrow p$.

We will see later when this converse is true.

Derived Forms of the Conditional:
2. The Inverse:

Let's again consider the conditional "If it rains, then I will drive you home." Define p and q and write, in words, the statement $\sim p \rightarrow \sim q$.

Definition: Consider the conditional $p \rightarrow q$. The **inverse** of the statement is $\sim p \rightarrow \sim q$.

When is this true?

Derived Forms of the Conditional:
3. The Contrapositive:

Let's again consider the conditional "If it rains, then I will drive you home." Define p and q and write, in words, the statement $\sim q \rightarrow \sim p$.

Definition: Consider the conditional $p \rightarrow q$. The **contrapositive** of the statement is $\sim q \rightarrow \sim p$.

When is this true?

Let's complete a truth table for all of these derived forms. I have started a table for us. (Make sure you understand it so far.) Let's complete it together.

		Negations		**Conditional**	**Converse**	**Inverse**	**Contrapositive**
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	T			
T	F	F	T	F			
F	T	T	F	T			
F	F	T	T	T			

So, which of these forms are equivalent?

The inverse is the contrapositive of the converse. What you talkin' 'bout, Willis?

expl 6: For each given statement, write it in symbols. Also, write the requested derived form in words.

a.) Statement: If you are a dog, then you will go to heaven. Find its contrapositive.

b.) Statement: You will be better informed if you read the newspaper. Find its converse.

c.) Statement: If yellow is *not* your favorite color, then I will give you my shirt. Find its inverse.