Pick two cards from a poker deck. What is the probability we get two Aces?

General Education Mathematics
Class Notes
Probability: Conditional Probability and Intersections of Events (Section 13.3)

In a three-child family, does the first two children being boys make it more or less likely that the third child is a boy? If we pull two cards out of a poker deck, does the first coming out red make it more or less likely that the second one is red too? This is the idea of independence which we explore here. We will also look at probabilities such as the sum of two dice is 5 *given* that one die is a 4.

Definition: Two events E and F are **independent** if the occurrence of event E in a probability experiment does *not* affect the probability of event F. Two events are **dependent** if the occurrence of E does, in fact, affect the probability of event F.

expl 1: Consider each experiment and pair of events. Which do you think are independent?

a.) Experiment: Pull two cards from a deck. Do *not* replace the first card before pulling the second.

Event A: First card is a face card

Event B: Second card is a face card

b.) Experiment: Roll two distinguishable, six-sided dice. One die is red; one is white.

Event A: Red die is 4

Event B: White die is an even number

c.) Experiment: Pull two cards from a deck. Replace the first card before pulling the second.

Event A: First card is a face card

Event B: Second card is a face card

d.) Experiment: Roll two distinguishable, six-sided dice. One die is red; one is white.

Event A: The red die is 6.

Event B: The sum of the two dice is 6.

If event A happens, does it change the likelihood of event B happening?

expl 2: We'll consider the experiment and events explored in example 1*b*. It is copied here.
Experiment: Roll two distinguishable, six-sided dice. One die is red; one is white.

Event A: Red die is 4

Event B: White die is an even number

a.) Find $P(A)$.

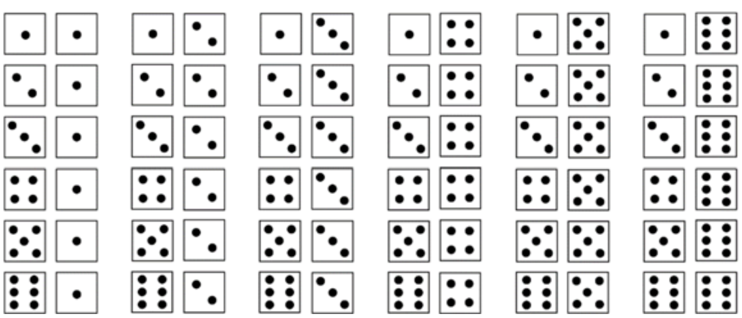
b.) Find $P(B)$.

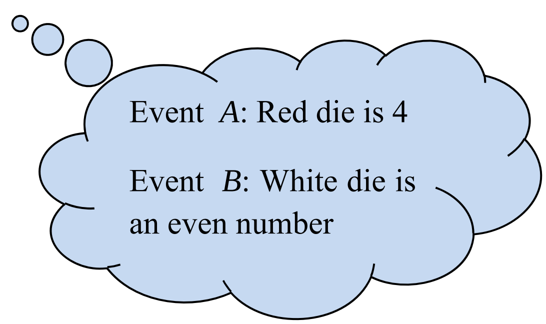
c.) Calculate $P(A) \cdot P(B)$.

What does AND mean?

When we talk about event E AND event F, we mean that *both* events occur simultaneously.

expl 2d: Now, returning to the previous experiment and considering the sample space given below, take the first die to be red and the second die to be white. Circle those outcomes that are in *both* events A *and* B.

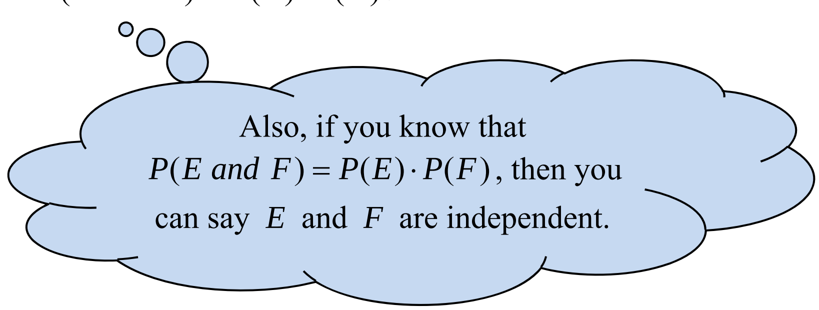




What is the probability $P(A \text{ AND } B)$? Do you notice anything about this value and what we got for example 2*c*? This leads us to a rule.

Multiplication Rule for Independent Events:

If E and F are independent events, then $P(E \text{ and } F) = P(E) \cdot P(F)$.



expl 3: Let's explore the probability of the dependent events seen in example 1*a*. It is copied here. What is the probability that both A and B occur, or $P(A \text{ AND } B)$?

Experiment: Pull two cards from a deck. Do *not* replace the first card before pulling the second.

Event A: First card is a face card

Event B: Second card is a face card

This is called *without* replacement.

We can do this two different ways.

Method 1: Using the Very Useful Formula for Probability:

Use combinations.

First, find the number of successes. How many ways can we choose 2 cards out of the 12 face cards?

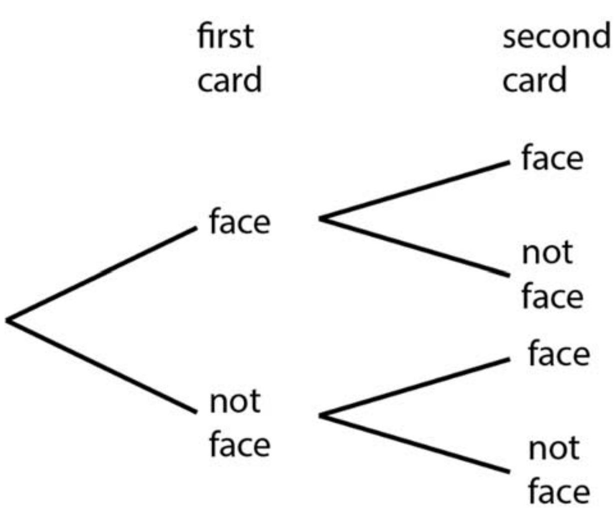
Next, find the number of possibilities. How many ways can we choose 2 out of the 52 total cards?

A poker deck contains four suits: diamonds, hearts, spades, and clubs. The diamonds and hearts are red and the spades and clubs are black. Each suit has thirteen cards: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, and King. This makes a total of 52 cards. A face card will be defined to be a Jack, Queen, or King. An even number means 2, 4, 6, 8, or 10.

Our Very Useful Formula is simply the number of successes divided by the number of possibilites, so find $P(1^{st} \text{ face AND } 2^{nd} \text{ face})$.

Method 2: Using a Tree Diagram:

Let's fill in the probabilities for this tree diagram together, using the number of successes divided by the number of possibilities. Mark the branch that is relevant to our question. You need only fill in this branch with probabilities.

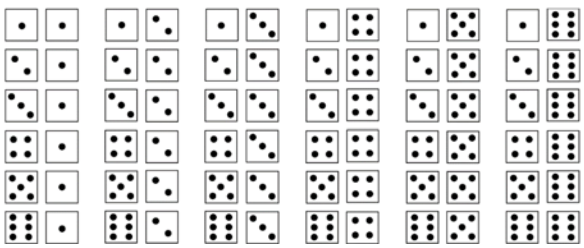


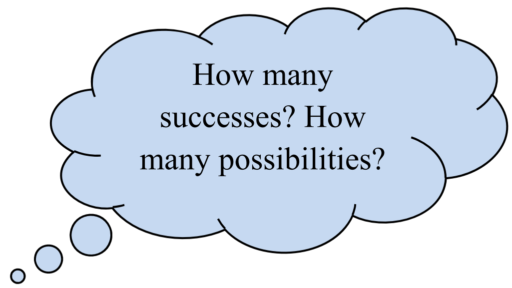
The probability of the second card being a face card *depends on whether or not* the first card was a face card.

We have some terminology and notation for situations similar to the one seen in example 3.

Definition: Conditional Probability: The notation $P(F \mid E)$ is read “the probability of event F given event E”. It is the probability that the event F occurs *given* that event E has occurred.

expl 4: Consider rolling two distinguishable, six-sided dice. Here is the sample space. Answer the questions that follow.

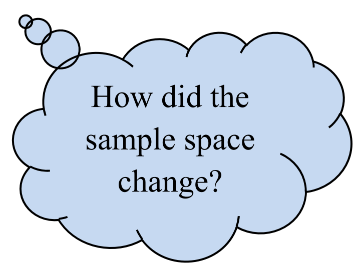




a.) What is the probability that the sum of the two dice is 5? Circle those successes.

b.) What is the probability that the first die is a 4? Circle those successes.

c.) *Given that the first die is a 4*, what is the probability that the sum is 5? What are the possibilities now? How many successes are there?

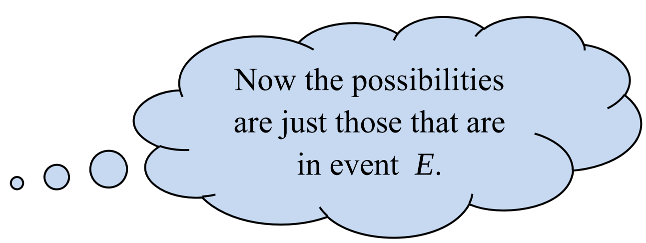


Conditional Probability Rule:

The fact that the first die is known to be 4 changes the probability that the sum is 5 because it reduces the number of possibilities we’ll consider. Only those outcomes where the first die is 4 are possibilities now. This takes our familiar formula which was always

$$\text{Probability of an event} = \frac{\text{number of successes}}{\text{number of total possibilities}}$$

and makes it more like $P(F \mid E) = \dfrac{N(E \text{ and } F)}{N(E)}$.



Another version of this formula, using the probabilities, tells us $P(F \mid E) = \dfrac{P(E \text{ and } F)}{P(E)}$.

Alternative Version of Conditional Probability Rule:
Another way to state this rule is $P(E \text{ and } F) = P(E) \cdot P(F \mid E)$. This is reminiscent of another formula we just saw on page 2 (for independent events), $P(E \text{ and } F) = P(E) \cdot P(F)$.

So, if the events are *not* independent, we use this formula with conditional probabilities.

expl 5: The data below show the marital status of males and females 15 years old or older in the US in 2013. Answer the following questions.

Marital Status	Males (in millions)	Females (in millions)	Total (in millions)
Never married	41.6	36.9	**78.5**
Married	64.4	63.1	**127.5**
Widowed	3.1	11.2	**14.3**
Divorced	11.0	14.4	**25.4**
Separated	2.4	3.2	**5.6**
Total	**122.5**	**128.8**	**251.3**

(Source: U.S. Census Bureau, Current Population Reports)

a.) What is the total number of males?

How many successes? How many possibilities?

b.) Find the probability that a person selected at random would be widowed *given* that the person is a male.

c.) Find the probability that a person selected at random would be a male *given* that the person is widowed.

How many successes? How many possibilities?

d.) Return to questions *b* and *c* to write a sentence for each that helps show the difference between what you found.

expl 6: Eliza has a 64% probability of making a free-throw in basketball practice. Assume all free-throws are independent of one another. Answer the following questions.

a.) What is the probability that Eliza makes three free-throws in a row?

b.) What is the probability that Eliza makes three free-throws in a row, but does *not* make four in a row?

This would look like "make, make, make, miss".

c.) If Eliza makes three attempts, what is the probability that she misses *at least* one?

This is the complement of the event in part *a*.

expl 7: Let's return, sort of, to example 3. However, this time, we *will replace* the card in the deck between draws. (In fact, these are the *independent* events explored in example 1*c*.)

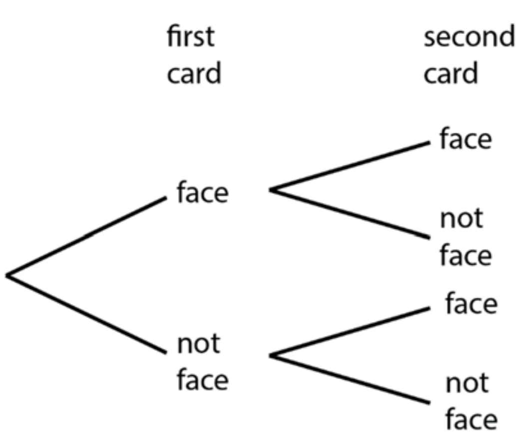
Experiment: Pull two cards from a deck. Replace the first card before pulling the second.

Event *A*: First card is a face card

Event *B*: Second card is a face card

This is called *with* replacement.

What is the probability that both *A* and *B* occur, or *P*(*A* AND *B*)? Let's fill in the probabilities for this tree diagram together, using the number of successes divided by the number of possibilities. Mark the branch that is relevant to our question. You need only fill in this branch with probabilities.



The probability of the second card being a face card *no longer depends* on whether or not the first card was a face card. Why?