Pick the right strategy and your job is easier. Also, let's see what George Polya thinks. (Who's that?)

General Education Mathematics
Class Notes
Problem Solving: Strategies and Principles (Section 1.1)

In the 1940's, a mathematician named George Polya developed this relatively simple and useful procedure for solving any story problem. It is probably somewhat familiar to you as it has been copied many times over the years. It is nice to see it from the horse's mouth, so here it is.

**George Polya's Method for Solving Problems:**

1. UNDERSTAND THE PROBLEM

    - **First.** You have to *understand* the problem.
    - What is the unknown? What are the data? What is the condition?
    - Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
    - Draw a figure. Introduce suitable notation.
    - Separate the various parts of the condition. Can you write them down?

2. DEVISING A PLAN

    - **Second.** Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.
    - Have you seen it before? Or have you seen the same problem in a slightly different form?
    - *Do you know a related problem?* Do you know a theorem that could be useful?
    - *Look at the unknown!* Try to think of a familiar problem having the same or a similar unknown.
    - *Here is a problem related to yours and solved before. Could you use it?* Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?
    - Could you restate the problem? Could you restate it still differently? Go back to definitions.

- If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or data, or both if necessary, so that the new unknown and the new data are nearer to each other?
- Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

3. CARRYING OUT THE PLAN

- **Third.** *Carry out* your plan.
- Carrying out your plan of the solution, *check each step*. Can you see clearly that the step is correct? Can you prove that it is correct?

Ooh! Check *each* step!

4. LOOKING BACK

- **Fourth.** *Examine* the solution obtained.
- Can you *check the result*? Can you check the argument?
- Can you derive the solution differently? Can you see it at a glance?

(source: *How To Solve It*, by George Polya, 1957)

Our textbook has many different strategies they want us to try out in the section. It may seem a bit overwhelming but take your time and try to write everything down as you work a problem. Some of the strategies are described by Polya such as drawing a picture or looking at a simpler problem to gain insight. We will explore some here through examples. I did rephrase some.

**Strategy: Draw a picture:**

expl 1: Four campers, Adliya, Benjamin, Christine, and Dari, have just arrived at the Seeds of Peace Camp in Maine for an orientation session. Each will shake hands with all of the others. Draw a picture to illustrate this situation, and determine the number of handshakes.

Use a single letter to represent each person. Connect each pair to represent each handshake.

**Strategy: Make a Systematic List:**

expl 2: Nico is considering which optional features to include with his new smartphone. He has narrowed it down to three choices: quick-charging battery, multiple windows, and a high-resolution camera lens. Depending on price, he will decide how many of these options he can afford. In how many ways can he make his decision?

Let's make a table to organize these options and whether or not he selects each one. Use a 0 to indicate he will *not* choose the option and a 1 to indicate he chooses the option. The first row is done for you.

| **quick-charging battery** | **multiple windows** | **high-resolution camera lens** |
|---|---|---|
| 1 | 0 | 0 |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |

The first row means he chooses the battery but neither other option. What else can happen? Try to keep your table organized.

**Strategy: Draw a Tree Diagram:**

expl 3: Draw a tree diagram to count the various results this game has.
*Roll a six-sided die and toss a coin.*

**Strategy: Use Patterns:**

expl 4: Complete the list of numbers. How do you know?

a.) 6, 10, 14, 18, 22, ______, ______, ______, ______

Find the differences between each two successive numbers. Perhaps, find the differences of the differences.

b.) 50, 40, 31, 23, ______, ______, ______, ______

**Strategy: Find a Counterexample:**

We can prove a statement is *not* true by finding a single counterexample.

expl 5: Show these operations are *not* equivalent by finding a counterexample.

*A: Triple a number, then add 5.*

*B: Add 5 to a number, then triple it.*

On the other hand, a single example does *not* prove a statement to be true, does it? Think about the algebraic statement $x^2 = 2x$.

Pick virtually any number and do what A and B say. Are the results equal?

**Strategy: Guess-and-Check:**

expl 6: Use guess-and-check to answer the question.
Mark invests \$1000 in two accounts for one year. The first account pays 5% simple interest and the second account pays 8% simple interest. If he made \$66.50 in interest, how much did he invest in each account?

(Hint: Simple interest is figured by the formula $I = PRT$, where $I$ is the interest earned, $P$ is the amount invested (principal), $R$ is the interest rate (in decimal form), and $T$ is the time in years.

This problem uses $T = 1$ to make it easier. If time is *not* mentioned, assume it to be one year.)

This is a typical "algebra" problem but it can be solved by guessing! Follow the table.

| **Amount in 5% account** | **Amount in 8% account (subtract)** | **Interest earned from 5% acct. (I = PR)** | **Interest earned from 8% acct. (I = PR)** | **Total interest (add)** | **Rating (Did we get \$66.50?)** |
|---|---|---|---|---|---|
| \$500 | \$1000 – 500 = \$500 | 500 · .05 = \$25 | 500 · .08 = \$40 | 25 + 40 = \$65 | No, \$65 is *too low*. |
| | | | | | |
| | | | | | |
| | | | | | |

We start with a guess in the first column. That allows us to calculate the rest. We got \$65 in total interest instead of \$66.50. What should our next guess be?

Do *not* forget to answer the question, in a full sentence preferably. How much did he invest in each account?

**Strategy: Split Hairs:**

It may seem at times that your math teacher is being overly exacting with a definition or concept. We do this because small differences can really make a difference. Be sure that you know *exactly* what symbols mean. Read problems carefully for details and do *not* assume information that is *not* given.

expl 7: For each, give the difference between the symbols or sentences. Do *not* worry if you do not know what the symbols mean; we'll get there.

a.) $<$ and $\leq$

b.) $=$ and $\approx$

c.) $4^3$ and $3^4$

d.) "Malia and Sasha will go to the store." and "Malia or Sasha will go to the store."

**Strategy: Relate a New Problem to an Old One:**

expl 8: Consider what was done in example 3 and expand on it here to count the number of results this game has.
*Roll a six-sided die and toss* two *coins.*

**Other Strategies:**

We will also want to keep in mind these other time-tested strategies.

-- Choose good names for variables so they are easy to remember (like $w$ for the width of a rectangle). It's still good form to specifically write down what variables represent.

-- Think carefully about the implied order of operations in sentences and expressions.

-- When facing a complicated problem, try simplifying certain aspects so you can focus on strategy. Then return to the original problem using your newfound insight.

-- Understand mathematical rules are *always* true. Math gets its power from this. For instance, a number raised to the power of 0 is always 1 (written $n^0 = 1$). Exceptions will *always* be noted as well; look for them. For instance, the rule is *not* true if $n$ is 0 (written $0^0 \neq 1$).

-- Often (but *not always*, ironically), math terms come from their English meanings. The term "slope" you see in linear equations can be understood if you think about the slope of a roof or ski slope.

-- Combining techniques is often quite useful, of course. The book talks about thinking about a problem in three ways: verbally (by restating it, perhaps), graphically (by drawing a picture or graph), and by example (by using actual numbers to make sense of it).

Problem solving is difficult at times, no doubt. Pushing forward and checking your logic along the way can help. Do *not* get discouraged; ask for help when you need it.

Remember practicing these skills will make you better and better. Good luck and enjoy the challenges ahead!

**Optional Worksheet: Polya's Problem Solving Techniques:**
This worksheet reiterates the information on pages 1 and 2 but also elaborates on it a bit.