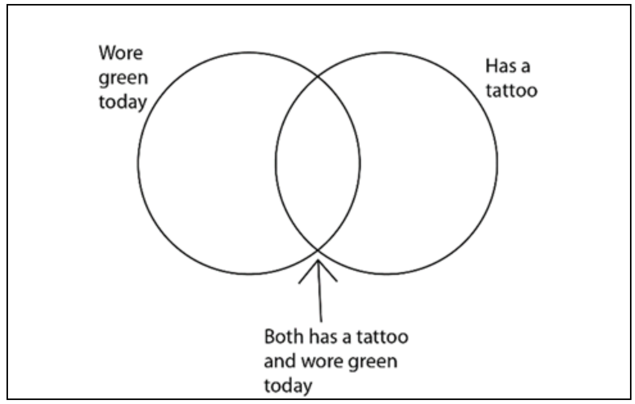
How do we subtract two sets? Can we add them? Where do two sets overlap? Can we do this with 3 sets?

General Education Mathematics
Class Notes
Set Operations (Section 2.3)

Let's consider this example of two overlapping sets shown in the last section.

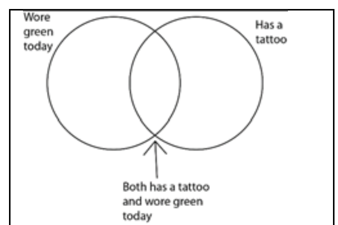
The left circle represents people who wore green that day. The right circle represents people who have a tattoo.

We define U, the universal set, as all of the students in my class.

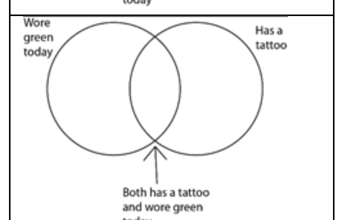



We will use this example to explore some important concepts in Set Theory. They are numbered below. Use the mini-Venn diagrams to shade each indicated area. Remember that U, the universal set, is all of the students in my class.

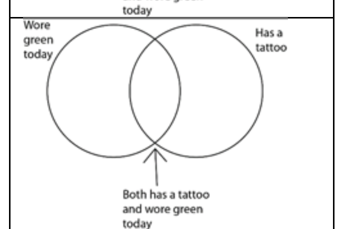
1. The area where the circles *overlap* is important. It is called the **intersection** of the two sets. It is described below the circles. Shade it. Think about this group of people and make sure that description makes sense to you.



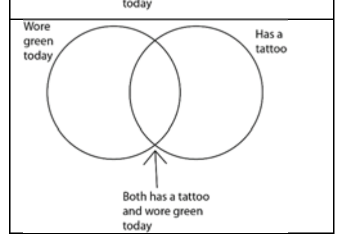

2. The area within either circle (or possibly both) is called the **union** of the two sets. Shade it. Can you describe the people in here?




3. The area outside of the left circle (but inside the rectangle because that is the "universe") is called the **complement** of that set. Shade it. Can you describe those people?




4. Now, think about the area that is inside the left circle but *not* inside the right circle. This is called the **difference** of the two sets. Shade it. Can you describe those people?



**Formal Definitions and Notation:**

Of course, there are more formal definitions for these notions. And, yes, there is notation.

**Definition:** The **intersection** of sets A and B, written $A \cap B$ is the set of elements that are in both A *and* B. In set-builder notation,
$A \cap B = \{x : x$ is a member of A *and* $x$ is a member of B$\}$.

The intersection of more than two sets is the set of elements that belong to every one of the sets.

We will use this notion for more than two sets.

**Definition:** If two sets have *no* elements in common, or $A \cap B = \varnothing$, then we say that A and B are **disjoint**.

**Definition:** The **union** of sets A and B, written $A \cup B$ is the set of elements that are in either A or B (or possibly both). In set-builder notation,
$A \cup B = \{x : x$ is a member of A or $x$ is a member of B$\}$.

The union of more than two sets is the set of all elements belonging to *at least one* of the sets.

We will use this notion for more than two sets.

**Definition:** The **complement** of set A is the set of elements (in the universal set U) that are *not* elements of A. This set is denoted by $A'$ or $\overline{A}$. In set-builder notation,
$A' = \{x : x$ is a member of U but $x$ is *not* a member of A$\}$.

**Definition:** The **difference** of sets B and A is the set of elements that are in B but *not* in A. This set is denoted by $B - A$. In set-builder notation,
$B - A = \{x : x$ is a member of B *and* $x$ is *not* a member of A$\}$.

Do you see why we use the common notion of subtraction here?

Do not forget to return to 2.1 Notes to fill in the notation reference sheet.

expl 1: Consider the sets defined on the right. Find the sets described below. Write your answers in listing notation.

a.) $C \cap D$

b.) $C \cup D$

c.) $C \cup A$

d.) $A - C$

e.) $B'$

f.) $B' - C$

> To do this one, we need to *first* find $B'$, don't we?

U = {0, 1, 2, 3, 4, …, 10, 20, 21, 22, 23}

A = {0, 1, 2, 3, 4, …, 10}

B = {0, 2, 4, 6, 8, 10}

C = {1, 2, 3, 4, 5}

D = {20, 21, 22, 23}

> In your mind, use <u>OR</u> in place of $\cup$ and <u>AND</u> in place of $\cap$.

**Order of Operations with Sets:**

Just like normal order of operations when you simplify something like $5^2 - 4\,(8 + 7) + 2$, we will see that order of operations matters when you evaluate sets like we did above in example 1*f*. Luckily, a common mantra, *work stuff in parentheses first*, helps us out. Also, if there is a complement involved, find it before going on. Let's practice.

expl 2: Consider the sets defined on the right. Find the set described below. Write your answers in listing notation.

$(C \cup A) \cap B'$

U = {0, 1, 2, 3, 4, …, 10, 20, 21, 22, 23}

A = {0, 1, 2, 3, 4, …, 10}

B = {0, 2, 4, 6, 8, 10}

C = {1, 2, 3, 4, 5}

expl 3: Consider the sets defined on the right. Find the sets described below. Write your answers in listing notation.

U = {azalea, bamboo, clover, daisy, elm, foxglove, goldenrod, holly, iris}

A = {azalea, bamboo, clover, daisy, elm}

B = {daisy, elm, foxglove, goldenrod}

Use letters to abbreviate the elements.

a.) $(A \cup B)'$

b.) $A' \cap B'$

Do you notice anything special about the answers to the above example?

**DeMorgan's Law:**

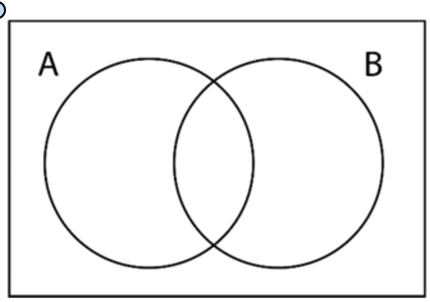
This is, in fact, an interesting truth called **DeMorgan's Law**. Actually, there are two related laws we will see. They are named after Augustus DeMorgan, a 19th-century British mathematician. Let's state them fully and then prove one using Venn diagrams.

**DeMorgan's Laws:** Consider two sets A and B. It is always true that $(A \cup B)' = A' \cap B'$. Also, $(A \cap B)' = A' \cup B'$ is always true.

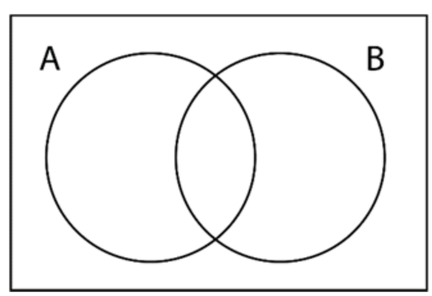
Do you see he just switched the symbols around?

Let's prove it. Use the Venn diagrams below to shade in the regions indicated. You will see that we really are talking about the same sets. (How cool!)

Venn diagrams can be used to deductively prove statements.




Shade in $(A \cap B)'$.



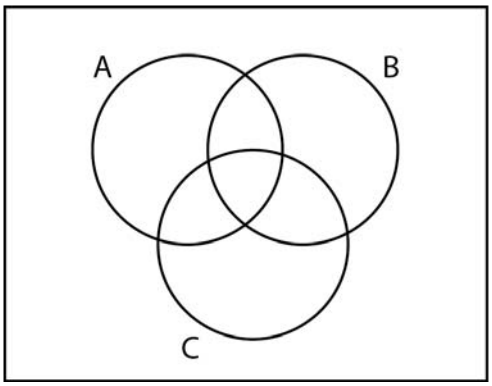

Shade in $A' \cup B'$.

expl 4: Use order of operations and Venn diagrams to evaluate if this statement is true. These statements involve three sets; notice how that Venn looks.

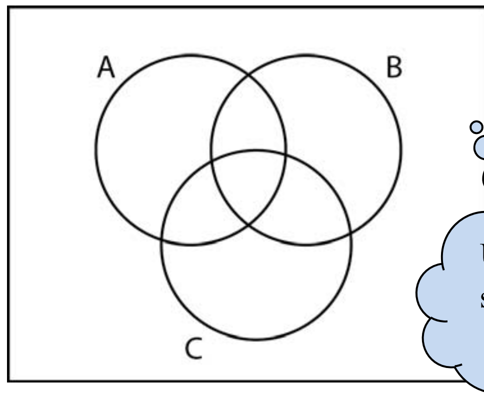
$$\left(A\cup\left(B\cup C\right)\right)' \stackrel{?}{=} \left(A\cap\left(B\cap C\right)\right)'$$

This "statement" says these two sets are equal. Are they? The question mark means we do *not* know.

Shade the Venn diagrams below with each side of the equation. Remember to think through the order of operations. Is this equation true?




Shade $\left(A\cup\left(B\cup C\right)\right)'$.




Shade $\left(A\cap\left(B\cap C\right)\right)'$.

Use pencil so you can erase.

**Inclusion-Exclusion Principle:**

Consider two sets, G and T. We have seen these two sets earlier but now we give them these convenient names. Let G be the set of students who wore green last Friday and let T be the set of students who had tattoos at that time.

Suppose we know there were 15 students wearing green, so $n(\mathrm{G}) = 15$.

Do you remember this notation?

At the same time, let's say $n(\mathrm{T}) = 12$.

In words, what does that mean?

Now, Bill concludes that there must be a total of 15 + 12 or 27 students who wore green or had a tattoo. Why is he wrong? What is he *not* considering?

Let's not blame him; it's a common mistake. So common that we have a Principle (yes, with a capital P) to address it.

In turns out that there are students who were counted in both sets G and T, those students with tattoos who *also* wore green.

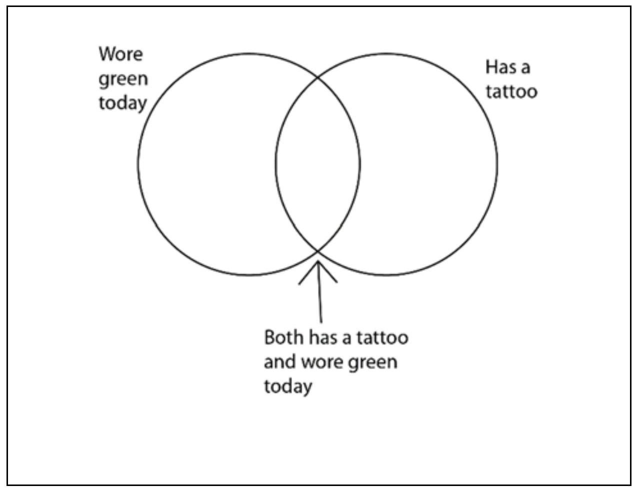
For the sake of argument, let's say there were 5 such students. In order to count the total number of students, we add 15 + 12 but then have to subtract that 5 to get 22.

(We are ignoring the other students who wore neither green nor had a tattoo because of their utter blandness. Seriously, what did they expect?)

To grok this idea, let's label the regions of the Venn diagram here with the numbers we have. Start with the 5 students with tattoos who *also* wore green. What region is that?

Since we know there were 15 students who wore green, how many wore green but were *not* tattooed? Label that region in the Venn.

Lastly, there were 12 tattooed students in total. How many had tattoos but did *not* wear green? Label it.




So, when Bill added 15 + 12 to get 27 students, he was essentially counting that intersection piece twice, wasn't he? The Inclusion-Exclusion Principle subtracts it off again to make it right.

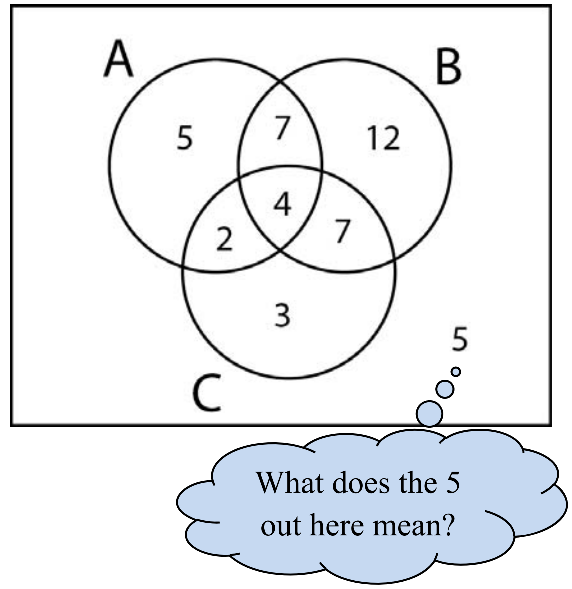
**Inclusion-Exclusion Principle:**

Consider two sets A and B. We know that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

$$n(A \underline{\text{ OR }} B) = n(A) + n(B) - n(A \underline{\text{ AND }} B)$$

We can also apply this concept to problems involving three sets. We may be given a Venn diagram with the number of elements in each region labeled. Let's look at that!

expl 5: Consider the Venn here with the number of elements in each region labeled. Find the following.




a.) $n(A)$

b.) $n(A \cup B)$

c.) $n(A \cap C)$

expl 6: Now, for the above sets, let's say A is the set of people who like chocolate donuts, B is the set of people who like crullers, and C is the set of people who like jelly donuts. Fill in the blanks with words to describe these sets.

a.) $A \cap C$ describes the people who like ________________________________________

b.) $C'$ describes the people who ________________________________________

c.) $A \cup B$ describes the people who like ________________________________________