We will draw approximate solutions curves for 1^{st}-order diff. eq. initial value problems.

Differential Equations
Class Notes
The Approximation Method of Euler (Section 1.4)

Definition: Euler's Method (or **tangent line method**): **Euler's Method** is a procedure for constructing approximate solutions to an initial value problem for a first-order diff. eq. $y' = f(x, y), \quad y(x_0) = y_0$.

It is a mechanical or computerized method of sketching a solution by hand using the direction field.

First, recognize that if we define $y' = f(x, y)$, then $f(x, y)$ is the slope of the solution curve y at the point (x, y) as before. Do not let the switch in our notation distract you.

Remember that f is the derivative of y with respect to x.

What we will do:

Start at (x_0, y_0) and find $f(x_0, y_0)$. This is the slope of y at this point so draw a tangent line segment (with that slope) until we get to a second point we'll call (x_1, y_1). Rinse and repeat.

So, we will next find $f(x_1, y_1)$ and use that to draw another tangent line segment to get to a third point (x_2, y_2). Repeat, repeat, repeat… Oh, bless the mighty computer! All bow to our computer overlords!

Our book illustrates this nicely.

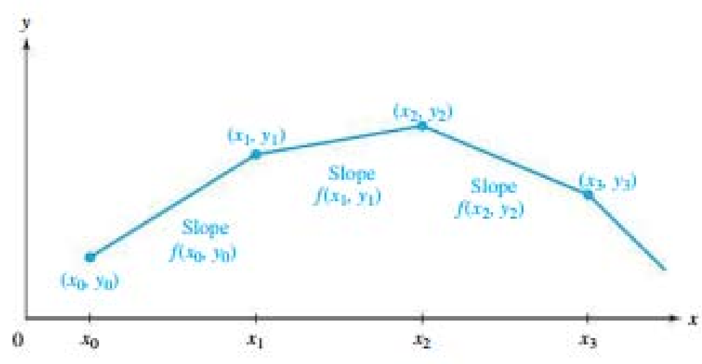


Figure 1.15 Polygonal-line approximation given by Euler's method

So how do we pick the points we use?
We start with (x_0, y_0) which is given to us. It is called the initial condition. You decide (or are told) the step size h to use. This **step size** is the difference between successive x-values.

Euler's Method Procedure:

For first-order diff. eq. $y' = f(x, y)$ with initial condition (x_0, y_0) and step size h, we use the following formulas.

The book shows the formulas' derivations.

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + h \cdot f(x_n, y_n) \quad \text{where } n = 0, 1, 2, \ldots$$

We will do this once by hand in class so you see the process slowly. However, I will expect you to use an online calculator for homework.

expl 1: Use Euler's method to approximate the solution to the following initial value problem at the points $x = 0.1, 0.2, 0.3, 0.4, 0.5$ (using step size $h = 0.1$). We will use the table below to organize.

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

Use the given formula for *dy/dx*.

To find the next x and y, use Euler's formulas.

$y(0) = 1$

x	***y***	***dy/dx***	**formula work**
$x_0 = 0$			
$x_1 = 0.1$			
$x_2 = 0.2$			
$x_3 = 0.3$			
$x_4 = 0.4$			
$x_5 = 0.5$		(do *not* need)	

Online Euler calculator:
We will use an online calculator. There is a link from www.stlmath.com but its direct address is www.math-cs.gordon.edu/~senning/desolver. You can find others on the Internet.

expl 2: Use Euler's method to find approximations to the solution of the initial value problem
$y' = 1 - \sin(y), \quad y(0) = 0$ at $x = \pi$ taking 1, 2, 4, and 8 steps.

They want $y(\pi)$.

We will play with N, the number of steps between x_0 and the final x-value.

First, what is (x_0, y_0)?

Our final x-value is always what?

What will our x-values be if we take 1, 2, 4, or 8 steps?

Number of steps, N	**The values of x, or x_i**	**Step size, h**
1		
2		
4		
8		

We will see this online calculator in class. The following should be kept in mind.

-- The online calculator uses the variables (t, y).
-- They use t_1 to mean the *final* t-value (or x-value, using the other assignment of variables).
-- You need to use multiplication signs and parentheses where applicable.
-- You can use "pi" for π.
-- Select "Graph and Data points" under "Output format".
-- Step size h (remember, *not* the same as N) needs to be a fraction or decimal. You *cannot* use "pi" but you can use 3.14 in its place.

-- The output is an approximate graph of the solution function y. The rudimentary table shows x and y-values along the way. The final y-value in the table is our desired y-value.

Do you see a segmented graph?

expl 2 continued: Record the various values of $y(\pi)$ here.

Step size, *h*	**Output value for $y(\pi)$**
π	
$\pi/2$	
$\pi/4$	
$\pi/8$	

Which value of $y(\pi)$ would you say was likely the most accurate approximation?

…But more steps bring more computations and hence greater roundoff error.

Finding a step size within an acceptable margin of error:

expl 3: Use the strategy of example 3 [in book] to find a value of h for Euler's method such that $y(1)$ is approximated to within ± 0.01, if $y(x)$ satisfies the initial value problem $y' = x - y, \quad y(0) = 0$.

Also, find, within ± 0.05, the value of x_0 such that $y(x_0) = 0.2$. Compare your answers with those given by the actual solution $y = e^{-x} + x - 1$.

Use the online calculator to fill in the table.

Plan: Repeatedly use Euler's method to approximate $y(1)$, halving h each time until the approximations are less than 0.01 apart. Use online calculator.

Number of steps	**Value of *h*, step size**	**Value given for $y(1)$ (Do *not* round.)**
2		
4		
8		
16		
32		
64		

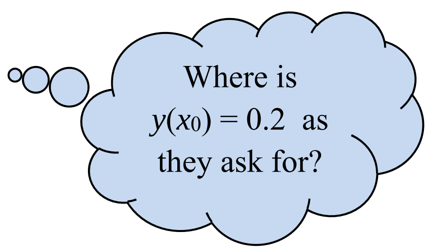
Round *final* $y(1)$ value to two decimal places. Include ± 0.01.

Quoting <u>Ordinary Differential Equations</u> (Tenenbaum, Pollard, 1985, p. 638): "A check on errors is to make all calculations over again with h half the size of the original h. If the result (from smaller h) agree with those from larger h to k decimal places, after being properly rounded off, then assume their common numerical value has k decimal place accuracy."

So, how close is our approximation? Use the given solution $y = e^{-x} + x - 1$ to find $y(1)$ to compare.

For the second question, we are asked to find, within ± 0.05, the value of x_0 such that $y(x_0) = 0.2$. We use h to be 0.1. That's twice the needed error of 0.05. You'll see why in a second. Use the online calculator to partially fill in the table.

x	***y***
0	0
0.1	
0.2	
0.3	
0.4	
0.5	
0.6	
0.7	
0.8	
0.9	
1.0	



Phrase your answer as a number rounded to two decimal places with ± 0.05. Check it against the actual solution.

expl 4: **Stefan's Law of Radiation:** This law states that the rate of change in temperature of a body at $T(t)$ kelvins in a medium at $M(t)$ kelvins is proportional to $M^4 - T^4$. That is,

$\frac{dT}{dt} = K\left(M(t)^4 - T(t)^4\right)$ where $K \in \mathbb{R}$. Let $K = 2.9 \text{ x } 10^{-10}$ (min)$^{-1}$ and assume that the medium temperature is constant, $M \equiv 293$ kelvins. If $T(0) = 360$ kelvins, use Euler's method with $h = 3.0$ minutes to approximate the temperature of the body after 30 minutes and 60 minutes.

Substitute K and M. Then write it as the online calculator will need it.

Use * for multiplication, including within scientific notation, and ^ for exponents.

Use $t_1 = 60$ and look through table to find $T(30)$ too.

Worksheet: Euler's Method for Approximating Function Values:

This worksheet practices Euler's method to approximate values of the unknown solution function. We also explore finding the value of the independent variable given the solution's value, to within a certain margin of error.