We solve systems of equations to find two related functions. We will also see initial value problems.

Differential Equations
Class Notes
Differential Operators and the Elimination Method for Systems (Section 5.2)

We will solve systems of linear differential equations with constant coefficients. Usually, we see two functions that are dependent on the variable t; often they are called $x(t)$ and $y(t)$.

Differential Operators:

Here, we recall to our minds the use of D to represent $\frac{d}{dt}$. As an example, let $x(t)=3t^2+4t$.

We might write $D[x]=6t+4$.

Here, $D[x]$ or Dx means $\frac{dx}{dt}$.

We assume the function is differentiable as many times as we need.

Dx

We might write $(D+2)[x]$ to mean $Dx+2x=\frac{dx}{dt}+2x=6t+4+2x$. We see here that you can distribute the $[x]$ to the parenthetical $(D+2)$. Some other properties of real numbers also apply as we will see.

expl 1: For $y=t^4+5$, find the following.

a.) $D[y]$ $= \frac{d}{dt}(y) = \frac{d}{dt}(t^4+5)$

$= 4t^3$

Don't leave y in there; substitute the expression in t.

b.) $(D-1)[y]$ $= D[y] - 1y$

$= 4t^3 - y = 4t^3 - t^4 - 5$

Conveniently, these operators work a lot like polynomials. We simply FOIL to find what $(D+3)(D-1)$ is equal to. Do it now.

$(D+3)(D-1) = D^2+2D-3$

So, $(D+3)(D-1)[y] = (D^2+2D-3)[y]$

$= y''+2y'-3y$

We can simplify things like
$(D+3)((D-1)[y])$ by first working the $(D-1)[y]$ part. Do that and then continue to interpret the $(D+3)$ part. Just use y as a generic function using primes to denote derivatives.

$$(D+3)((D-1)[y])$$
$$=(D+3)[y'-y]$$
$$=D[y'-y]+3[y'-y]$$
$$=y''-y'+3y'-3y=y''+2y'-3y$$

What did you get? Does it match what you got on the bottom of page 1?

What is D^2?

As we saw above, in fact, $D(D[x])$ means to take the derivative of the derivative (a composition). We will use D^2 to represent the **second derivative** with respect to t.

This idea and how we can use this with our algebraic rules still blows my mind a little bit.

expl 2: Show that $(D-1)((D+3)[y])$ is the same as we got above. What property does that reveal about this operator?

$$(D-1)((D+3)[y])$$
$$=(D-1)(D[y]+3y)$$
$$=(D-1)[y'+3y]$$
$$=D[y'+3y]-1(y'+3y)$$
$$=y''+3y'-y'-3y=y''+2y'-3y \quad \text{same}$$

So, $(D-1)(D+3)=(D+3)(D-1)$

Commutativity

What this means is that we can treat aD^2+bD+c as if it was an ordinary polynomial, as long as a, b, and c are *constants*. When variables are involved, this does *not* work.

expl 3: If we throw a variable into the mix, stuff goes sideways. Take y to be a function in t. Show that $(D+3t)(D[y])$ is *not* the same as $D((D+3t)[y])$.

We see $(D+3t)(D[y])$ as $(D+3t)[y']=D[y']+3ty'$.

$$(D+3t)(D[y])=(D+3t)[y']$$
$$=D[y']+3ty'$$
$$=y''+3ty'$$

Hey, these aren't the same!

$$\text{But } D((D+3t)[y])=D[D[y]+3ty]$$
$$=D[y'+3ty]$$
$$=D[y']+D[3ty]=y''+3y+3ty'$$

5.2

Solving Systems of Linear Differential Equations with Constant Coefficients:
What we will do in practice is rewrite a system of equations using this derivative operator D. We will then solve it using a procedure that is strikingly similar to the procedure used for ordinary linear systems of equations.

Our method will result in homogeneous and nonhomogeneous linear equations as we solved in the last chapter. So, get those Notes out and let's get going!

Elimination Method for 2x2 Systems:

To find the general solution to $\begin{array}{l} L_1[x]+L_2[y]=f_1 \\ L_3[x]+L_4[y]=f_2 \end{array}$ where L_i is a polynomial in $D = d/dt$, do the following.

a.) Be sure the system is in operator form.
b.) Eliminate one variable – say, y – and solve for the other variable – say $x(t)$. This will likely involve our methods for homogeneous and nonhomogeneous equations. If the system is **degenerate** – stop! A separate analysis is needed; more on that later.
c.) Shortcut: If possible, use the system to derive an equation that involves the variable you eliminated from the last step (probably $y(t)$) but *not* its derivatives. (Otherwise, go to step *d*.) Substitute the found expression for $x(t)$ into this equation to get the $y(t)$ formula. You will have some constants and so have a *general* solution for $x(t)$ and $y(t)$.
d.) Eliminate the variable you left behind in step *b* (probably x), solving for the other. Again, the methods for homogeneous and nonhomogeneous equations will be needed. This will yield twice as many constants as are needed; go to step *e*.
e.) Remove extra constants by subbing $x(t)$ and $y(t)$ into one or both equations in system. Write x and y in terms of the remaining constants to obtain the *general* solution.

Hey buddy, who you callin' a *degenerate*?
Calm down, don't take it personally. It's just your nature.

Definition: Degenerate system: For the system $\begin{array}{l} L_1[x]+L_2[y]=f_1 \\ L_3[x]+L_4[y]=f_2 \end{array}$, the system is said to be **degenerate** if and only if $L_1L_4 - L_2L_3 = 0$.

Such a system will have either *no* solutions or an *infinite* number of solutions.

expl 4: Find a general solution to the linear system. Differentiation is with respect to t.

$x' = x - y$

$y' = y - 4x$

Once it's in D form, how do you eliminate one variable? You'll solve the resulting equation by chapter 4 methods.

@ $x' - x + y = 0 \rightarrow (D-1)[x] + y = 0$

$4x + y' - y = 0 \quad 4x + (D-1)[y] = 0$

$(D-1)\big((D-1)[x] + y = 0\big)$

$-1\big(4x + (D-1)[y] = 0\big)$

$(D^2-2D+1)[x] + \cancel{(D-1)[y]} = 0$ add

$-4x - \cancel{(D-1)[y]} = 0$

$(D^2-2D-3)[x] = 0$

$x'' - 2x' - 3x = 0$

aux eqn $r^2 - 2r - 3 = 0$

$(r+1)(r-3) = 0$

$r = -1, 3$

(Section 4.2)

$\boxed{x = c_1 e^{-t} + c_2 e^{3t}}$

$x' = -c_1 e^{-t} + 3c_2 e^{3t}$

ⓒ $x' = x - y$

$y = x - x'$

$y = c_1 e^{-t} + c_2 e^{3t} + c_1 e^{-t} - 3c_2 e^{3t}$

$\boxed{y = 2c_1 e^{-t} - 2c_2 e^{3t}}$

Depending on which variable you solve for first, your constants will look different (but be equivalent) to the book's solution. MML will specify a starting variable.

expl 5: Find a general solution to the linear system. Differentiation is with respect to t.

$x' + y' + 2x = 0$

$x' + y' - x - y = \sin t$

ⓐ

$(D-1)\,\big((D+2)[x] + Dy = 0\big)$

$-D\,\big((D-1)[x] + (D-1)[y] = \sin t\big)$

add

To find x: Section 4.2 will get us a general solution. Section 4.4 will get us a particular solution. Superposition says we can combine them.

ⓑ $(D^2+D-2)[x] + \cancel{(D-1)D[y]} = 0$

$(-D^2+D)[x] - \cancel{D(D-1)[y]} = -D[\sin t]$

$(2D-2)[x] = -\cos t$

$2x' - 2x = -\cos t$

aux eqn: $2r - 2 = 0$

$2r = 2$

$r = 1$

$\binom{\text{sect}}{4.2}$ $x_g = C_1 e^t$

Section 4.4: $m = 0,\ \alpha = 0,\ \beta = 1$

$\rightarrow s = 0$

$x_p = A\cos t + B\sin t$

$x_p' = -A\sin t + B\cos t$

$2x' - 2x = -\cos t$

$-2A\sin t + 2B\cos t - 2A\cos t - 2B\sin t = -\cos t\ \{+0\sin t$

$-2A - 2B = 0$

$-2A + 2B = -1$ add

$-4A = -1$

$A = 1/4$

$-2A - 2B = 0$

$-2(1/4) - 2B = 0$

$-1/2 - 2B = 0$

$-2B = 1/2$

$B = -1/4$

$$\boxed{x_p = \frac{1}{4}\cos t - \frac{1}{4}\sin t}$$

Particular soln

The shortcut (step *c*) is doable. The alternative, step *d*, *would* have us *start over* with equations to eliminate the second variable.

So, section 4.5 (superposition) gives us

$$\boxed{x = C_1 e^t + \frac{1}{4}\cos t - \frac{1}{4}\sin t}$$

(extra room for work)

c $x' + y' + 2x = 0$

$-1(x' + y' - x - y = \sin t)$

$x' + y' + 2x = 0$

$-x' - y' + x + y = -\sin t$ add

$3x + y = -\sin t$

$y = -\sin t - 3x$

$y = -\sin t - 3c_1 e^t - \frac{3}{4}\cos t + \frac{3}{4}\sin t$

$$\boxed{y = \frac{-1}{4}\sin t - \frac{3}{4}\cos t - 3c_1 e^t}$$

$$\boxed{\text{and } x = c_1 e^t + \frac{1}{4}\cos t - \frac{1}{4}\sin t}$$

To find y through step *d*: Section 4.2 will get us a general solution. Section 4.5 will get us a particular solution. Superposition says we can combine them.

But that's unnecessary since we will do the shortcut in step *c*.

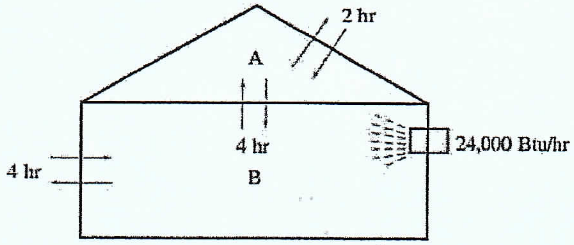
Initial Value Problems: If given, initial values for x and y will enable us to solve for the constants in the general equations.

Diff eq is hard but algebra is harder!

for zone B

$$u(t) = \frac{24{,}000 \cancel{\text{Btu}}}{1 \text{ hr}} \cdot \frac{\frac{1}{2}\,°F}{1000 \cancel{\text{Btu}}} = 12\ °F/hr$$

expl 6: A home has two zones, an attic (zone A) and a lower, living space (zone B). The living space is cooled by an AC unit which removes 24,000 Btu/hr. The heat capacity of zone B is ½ °F per thousand Btu. The time constant for heat transfer between zone A and the outside is 2 hours (meaning K from our heating and cooling section is ½).



This time constant between B and the outside is 4 hours ($K = ¼$) and between zones A and B is 4 hours ($K = ¼$). If the outside temperature is a constant 100 °F, how warm will it eventually get in the attic? (zone a)

Let $a(t)$ and $b(t)$ represent the temperatures of the two zones at time t.

We know Newton's Law of Cooling to be $\frac{dT}{dt} = K\left[M(t) - T(t)\right]$. Use $M(t) = 100$.

The total rate of change in temperature for zone A would rely on the outside and zone B. The total rate of change in temperature for zone B would rely on the outside, zone A, and the AC unit.

$u(t)$

zone a:

$$\frac{da}{dt} = \frac{1}{2}(100 - a(t)) + \frac{1}{4}(b(t) - a(t))$$

(multiply by 4)

zone b:

$$\frac{db}{dt} = \frac{1}{4}(100 - b(t)) + \frac{1}{4}(a(t) - b(t)) - 12$$

(multiply by 4)

$$4a' = 200 - 2a + b - a \qquad 4b' = 100 - b + a - b - 48$$

$$4a' + 3a - b = 200 \qquad -a + 4b' + 2b = 52$$

a

$$(4D+3)[a] - b = 200$$
$$-a + (4D+2)[b] = 52$$

$$(4D+2)\big((4D+3)[a] - b = 200\big)$$
$$1\,\big(-a + (4D+2)[b] = 52\big)$$

$$(16D^2 + 20D + 6)[a] - (4D+2)[b] = (4D+2)[200]$$
$$-a + (4D+2)[b] = 52$$

$\underline{5.2}$

(extra room for work)

$$(16D^2+20D+5)[a] = 452$$

$$16a'' + 20a' + 5a = 452$$

aux eqn: $16r^2 + 20r + 5 = 0$ (section 4.2)

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-20 \pm \sqrt{400-4(16)(5)}}{32}$$

r_1, r_2

$$= \frac{-20 \pm \sqrt{80}}{32} = \frac{-20 \pm 4\sqrt{5}}{32} = \frac{-5 \pm \sqrt{5}}{8} = \frac{-5}{8} \pm \frac{\sqrt{5}}{8}$$

$$a_g = c_1 e^{\left(-\frac{5}{8}+\frac{\sqrt{5}}{8}\right)t} + c_2 e^{\left(-\frac{5}{8}-\frac{\sqrt{5}}{8}\right)t}$$

$$a_p = A, \quad A \in \mathbb{R}$$

$$a = \underbrace{c_1 e^{\left(-\frac{5}{8}+\frac{\sqrt{5}}{8}\right)t}}_{\to 0} + \underbrace{c_2 e^{\left(-\frac{5}{8}-\frac{\sqrt{5}}{8}\right)t}}_{\to 0} + \underline{\underline{A}}$$

What will a be as time goes on and on and on? (as $t \to \infty$). As $t \to \infty$, $a \to A$.

$a_p = A$

$a_p' = 0$

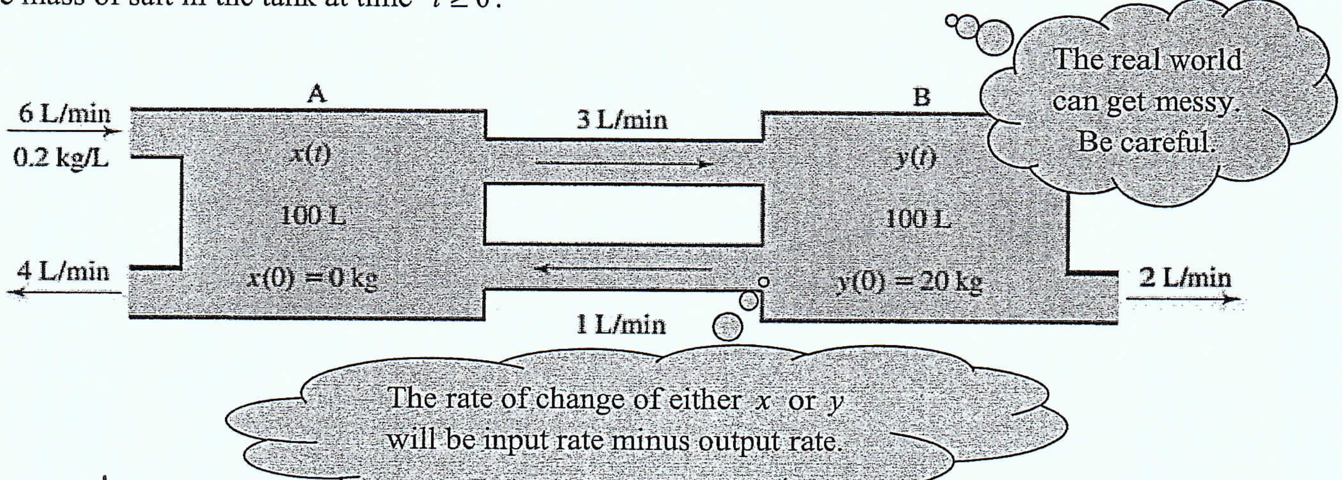
$a_p'' = 0$

$$16a'' + 20a' + 5a = 452$$
$$0 + 0 + 5A = 452$$
$$A = 90.4$$

So, as $t \to \infty$, the temp of the attic approaches $90.4°$ F.

(#31 in 5.2 hmwk)

Optional expl 7: Two large tanks, each holding 100 L of liquid, are interconnected by pipes. The liquid flows between them according to the picture here. The liquids inside the tanks are kept well stirred. At the left, we see an inlet pipe that delivers a brine solution at a concentration of 0.2 kg/L of salt. If initially tank A contains no salt and tank B contains 20 kg of salt, determine the mass of salt in the tank at time $t \geq 0$.



$$\frac{dx}{dt} = \text{input rate} - \text{output rate}$$

$$= \frac{6L}{min} \cdot \frac{0.2kg}{L} + \frac{1L}{min} \cdot \frac{y(t)}{100L} - \frac{3L}{min} \cdot \frac{x}{100L} - \frac{4L}{min} \cdot \frac{x}{100}$$

$$\frac{dx}{dt} = 1.2 + \frac{y}{100} - \frac{7x}{100}$$

Likewise, $\frac{dy}{dt} = \text{input rate} - \text{output rate}$

$$= \frac{3L}{min} \cdot \frac{x}{100} - \frac{1L}{min} \cdot \frac{y}{100} - \frac{2L}{min} \cdot \frac{y}{100L}$$

$$dy/dt = \frac{3x}{100} - \frac{3y}{100}$$

System: ⓐ $(D+0.07)[x] - 0.01y = 1.2$

$-0.03x + (D+0.03)[y] = 0$

$\begin{pmatrix} x(0)=0 \\ y(0)=20 \end{pmatrix}$

ⓑ solve: (multiply 1st by $100(D+0.03)$ or $(100D+3)$)

$(100D+3)((D+0.07)[x] - 0.01y = 1.2)$

$-0.03x + (D+0.03)[y] = 0$

add

$(100D^2+10D+0.21)[x] - (D+0.03)[y] = 3.6$

$-0.03x + (D+0.03)[y] = 0$

$$(100D^2 + 10D + 0.18)[x] = 3.6$$
$$100x'' + 10x' + 0.18x = 3.6$$

(extra room for work) (Section 4.4)

aux eqn $100r^2 + 10r + 0.18 = 0$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-10 \pm \sqrt{100-4(100)(0.18)}}{200}$$

$$= \frac{-10 \pm \sqrt{28}}{200} = \frac{-10 \pm 2\sqrt{7}}{200} = \frac{-1}{20} \pm \frac{\sqrt{7}}{100}$$

Section 4.4 $(s=0) \rightarrow x_p = A_0$, $x' = 0$, $x'' = 0$

$$100x'' + 10x' + 0.18x = 3.6$$
$$0 + 0 + 0.18A_0 = 3.6$$
$$A_0 = 20$$

So, $x_p = 20$ from section 4.4

and (section 4.2) $x_g = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

where $r_1 = \frac{-1}{20} + \frac{\sqrt{7}}{100} \approx -0.02354$ (from above)

and $r_2 = \frac{-1}{20} - \frac{\sqrt{7}}{100} \approx -0.07646$

$$\boxed{\text{So, } x = 20 + c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

(c) Now find y (shortcut):

(from dx/dt at beginning)

$$x' = 1.2 + 0.01y - 0.07x$$
$$0.01y = x' - 1.2 + 0.07x$$
$$y = 100x' - 120 - 7x$$

(continuation of #7)

$$y = 100x' - 120 - 7x$$

Using x found earlier,

$$y = 100(r_1 c_1 e^{r_1 t} + r_2 c_2 e^{r_2 t}) - 120$$
$$+ 7(20 + c_1 e^{r_1 t} + c_2 e^{r_2 t})$$

$$\boxed{y = (100r_1 + 7)c_1 e^{r_1 t} + (100r_2 + 7)c_2 e^{r_2 t} + 20}$$

Now, find c_1 and c_2 $(x(0)=0,\ y(0)=20)$:

$x(0)=0 \rightarrow \quad x = 20 + c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$$0 = 20 + c_1 e^0 + c_2 e^0$$

$$0 = 20 + c_1 + c_2$$

$$c_1 = -20 - c_2$$

$y(0)=20 \rightarrow \quad y = (100r_1 + 7)c_1 e^{r_1 t} + (100r_2 + 7)c_2 e^{r_2 t} + 20$

$$20 = (100r_1 + 7)c_1 \cancel{e^0}^{1} + (100r_2 + 7)c_2 \cancel{e^0}^{1} + 20$$

$$20 = (100r_1 + 7)(-20 - c_2) + (100r_2 + 7)c_2 + 20$$

$$\cancel{20} = -2000r_1 - 140 - 100r_1 c_2 - \cancel{7c_2} + 100r_2 c_2 + \cancel{7c_2} + \cancel{20}$$

$$2000r_1 + 140 = c_2(-100r_1 + 100r_2)$$ (r_1, r_2 defined earlier)

$$c_2 \approx -17.5593$$

$$c_1 \approx -37.5593$$

So,

$$x = 20 - 37.5593\, e^{r_1 t} - 17.5593\, e^{r_2 t}$$

and $y = -37.5593\,(100 r_1 + 7)\, e^{r_1 t}$

$\quad - 17.5593\,(100 r_2 + 7)\, e^{r_2 t} + 20$

where $r_1 = \frac{-1}{20} + \frac{\sqrt{7}}{100}$

and $r_2 = \frac{-1}{20} - \frac{\sqrt{7}}{100}$.