We have studied lower-order diff. eqs. when the coefficients were constants. Now we play with variable coefficients and higher orders.

Differential Equations
Class Notes
Basic Theory of Linear Differential Equations (Higher-Order Including Variable Coefficients)
(Section 6.1)

Definition: A **linear differential equation of order n** is of the form $a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+...+a_0(x)\cdot y(x)=b(x)$. Here, $a_0(x)$, ... $a_n(x)$ and $b(x)$ depend only on x, not y. The **standard form** of a linear diff. eq. is obtained by dividing all terms by $a_n(x)$ to get $\boxed{y^{(n)}(x)+p_1(x)y^{(n-1)}(x)+...+p_n(x)\cdot y(x)=g(x).}$

We assume that the functions $a_i(x)$, $b(x)$, and $g(x)$ are all continuous on some interval I and $a_n(x)\neq 0$ on I. We will further define the functions $p_i(x)$ to be continuous on I.

We will use our old friends, **homogeneous** and **nonhomogeneous**, to indicate if $b(x)$ (and therefore $g(x)$) is equal to 0 or *not*, respectively.

If these $a_i(x)$ functions are constants, we say the diff. eq. has **constant coefficients**. Otherwise, we have a diff. eq. with **variable coefficients.**

Theorem 1: Existence and Uniqueness for Initial Value Problems:
Suppose the functions $p_i(x)$ and $g(x)$ are all continuous on some interval (a, b) that contains the point x_0. Then, for any choice of the initial values $\gamma_0, \gamma_1, ...\gamma_{n-1}$, there exists a unique solution $y(x)$ on the whole interval (a, b) to the initial value problem $\boxed{y^{(n)}(x)+p_1(x)y^{(n-1)}(x)+...+p_n(x)\cdot y(x)=g(x)}$ with initial values $\boxed{y(x_0)=\gamma_0,\quad y'(x_0)=\gamma_1,\quad y''(x_0)=\gamma_2,\ ...\quad y^{(n-1)}(x_0)=\gamma_{n-1}.}$

Those are gamma.

We will be given values for all derivatives up through the $(n-1)^{\text{th}}$ at some x-value.

expl 1: Determine the largest interval (a, b) for which Theorem 1 guarantees the existence of a unique solution.

$x_0=\pi$

$\boxed{y'''-\sqrt{x}y=\sin x,}$ $y(\pi)=0,\quad y'(\pi)=11,\quad y''(\pi)=3$

Determine functions $p_i(x)$ and $g(x)$. Over what interval (containing π) are they all continuous?

$p_1(x)=0$
$p_2(x)=0$

$g(x)=\sin x$ (cont on $(-\infty,\infty)$)
$p_3(x)=-\sqrt{x}$ (cont on $[0,\infty)$)

Largest interval : $(0,\infty)$

Determine functions $p_i(x)$ and $g(x)$. Over what interval (containing π) are they all continuous?

expl 2: Determine the largest interval (a, b) for which Theorem 1 guarantees the existence of a unique solution.

$x_0 = 5$

$y''' - y'' + \sqrt{x-1} \cdot y = \tan x, \quad y(5) = y'(5) = y''(5) = 1$

$p_2(x) = 0$

$g(x) = \tan x$ (Not defined (not cont.) for $\frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \frac{7}{2}\pi$

$p_3(x) = \sqrt{x-1}$ $(x-1 \geq 0 \rightarrow x \geq 1)$

$p_1(x) = -1$ (cont $(-\infty, \infty)$)

largest interval (a,b) for which Thm 1 guarantees the existence of a unique soln : $\left(\frac{3}{2}\pi, \frac{5}{2}\pi\right)$

Definition: Differential Operator $L[y]$:

We will define the differential operator L as

$L[y] = \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + ... + p_n \cdot y = \left(D^n + p_1 D^{n-1} + ... + p_n\right)[y]$. Notice we can then express the standard form of our linear diff. eq. as $L[y]\,(x) = g(x)$.

Now, this differential operator L is linear (as can be proven but is *not* here). That means that $L[y_1 + y_2 + ... + y_m] = L[y_1] + L[y_2] + ... + L[y_m]$ and also $L[c \cdot y] = c \cdot L[y]$ for any constant c.

We will use this notation later.

Definition: Wronskian:

Let $f_1, \ldots, f_n$ be any n functions that are $(n-1)$ times differentiable. The function

$$W[f_1, f_2, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$ is called the

Wronskian of $f_1, \ldots, f_n$.

determinant

matrix

$[\quad]$

$|$ matrix's determinant

Online Determinant Calculator:
We will need to calculate some rather complex determinants. Rather than do that by hand (and drive ourselves to madness), we will use an online calculator. The good people at WolframAlpha have provided us with www.wolframalpha.com/calculators/determinant-calculator.

Theorem 2: Representation of Solutions (Homogeneous Case):
Let $y_1, \ldots, y_n$ be n solutions on (a, b) of the diff. eq. $L[y](x) = 0$ (otherwise known as $y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \ldots + p_n(x) \cdot y(x) = 0$) where the functions $p_i(x)$ are all continuous on some interval (a, b).

If at some point x_0 in (a, b) these solutions satisfy $W[y_1, \ldots, y_n](x_0) \neq 0$, then every solution of the diff. eq. on (a, b) can be expressed in the form $y(x) = C_1 y_1(x) + \ldots + C_n y_n(x)$ where $C_1, \ldots C_n$ are constants. (Proof given in book.)

Really, this is a generalization of an earlier definition.

This is the **general solution** to the diff. eq.

Recall: Definition: Linear Dependence of Functions:
The m functions $f_1, \ldots, f_m$ are said to be **linearly dependent on an interval** ***I*** if at least one of them can be expressed as a linear combination of the others on I. Equivalently, they are **linearly dependent** if there exists constants $c_1, c_2, \ldots, c_m$, *not all zero*, such that
$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_m f_m(x) = 0$ for all x in I.

Otherwise, they are said to be **linearly independent**.

It is helpful to remember that the following sets of functions are *always* linearly independent on every open interval (a, b). Add to this list as you see fit.

$\{1, x, x^2, x^3, \ldots x^n\}$

$\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots \cos(nx), \sin(nx)\}$

$\{e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots e^{\alpha_n x}\}$ where α_i's are distinct constants expl 3: Show that the functions are linearly *dependent* on the interval given. As proof, give values of the constants (*not all zero*) so that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$.

$\{\overset{f_1}{\sin^2 x},\ \overset{f_2}{\cos^2 x},\ \overset{f_3}{1}\}\quad (-\infty, \infty)$

we know $\sin^2 x + \cos^2 x = 1$

So, $\sin^2 x + \cos^2 x - 1 = 0$.

which is $1 \cdot \sin^2 x + 1 \cdot \cos^2 x + -1 \cdot 1 = 0$

$1 f_1 + 1 f_2 + -1 \cdot f_3 = 0$

and so, we found $c_1 = 1$, $c_2 = 1$, $c_3 = -1$ to be the constants needed (not all zero).

expl 4: Show that the functions are linearly *independent* on the interval given. In other words, show that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ implies that all three constants are indeed zero.

$\{x,\ x^{-1},\ x^{1/2}\}\quad (0, \infty)$

Pick three x-values in the interval to substitute. Solve the system with a matrix on the calculator.

So, $f_1(x) = x$

$f_2(x) = x^{-1}$

$f_3(x) = x^{1/2}$

$$
\begin{array}{l|l}
 & c_1 f_1 + c_2 f_2 + c_3 f_3 = 0 \\
\hline
x = 3 & 3c_1 + \frac{1}{3}c_2 + \sqrt{3}\,c_3 = 0 \\
x = 7 & 7c_1 + \frac{1}{7}c_2 + \sqrt{7}\,c_3 = 0 \\
x = 12 & 12c_1 + \frac{1}{12}c_2 + \sqrt{12}\,c_3 = 0
\end{array}
\rightarrow
\left[\begin{array}{ccc|c}
3 & \frac{1}{3} & \sqrt{3} & 0 \\
7 & \frac{1}{7} & \sqrt{7} & 0 \\
12 & \frac{1}{12} & \sqrt{12} & 0
\end{array}\right]
$$

$\downarrow$ solve by matrix on calculator

$c_1 = 0$, $c_2 = 0$, $c_3 = 0$

$\therefore$ The fncs are not linearly dependent.

Linear Dependence and the Wronskian:

If $y_1, \ldots, y_n$ are n solutions to the diff. eq. $y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \ldots + p_n(x) \cdot y(x) = 0$ on the interval (a, b) and where the functions $p_i(x)$ are all continuous, then the following statements are equivalent.

> i.) $y_1, \ldots, y_n$ are linearly *dependent* on (a, b).
> ii.) The Wronskian $W[y_1, \ldots, y_n](x_0)$ is zero at some point x_0 in (a, b).
> iii.) The Wronskian $W[y_1, \ldots, y_n](x)$ is identically zero on (a, b).

The following contrapositives of these statements are also equivalent.

> iv.) $y_1, \ldots, y_n$ are linearly *independent* on (a, b).
> v.) The Wronskian $W[y_1, \ldots, y_n](x_0)$ is *nonzero* at some point x_0 in (a, b).
> vi.) The Wronskian $W[y_1, \ldots, y_n](x)$ is *never zero* on (a, b).

Whenever statement iv, v, *or* vi is met, then we call $\{y_1, \ldots, y_n\}$ a **fundamental solution set** for the diff. eq. on (a, b).

Theorem 4: Representation of Solutions (Nonhomogeneous Case):

Let $y_p(x)$ be a *particular* solution to the *nonhomogeneous* equation $y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \ldots + p_n(x) \cdot y(x) = g(x)$ on the interval (a, b) where the functions $p_i(x)$ are all continuous on (a, b). Let $\{y_1, \ldots, y_n\}$ be a fundamental solution set for the corresponding homogeneous equation $y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \ldots + p_n(x) \cdot y(x) = 0$.

Then every solution of the nonhomogeneous eqn. on the interval (a, b) can be expressed in the form $y(x) = y_p(x) + C_1y_1(x) + \ldots + C_ny_n(x)$. (Proof given in book.)

This is the **general solution** to the diff. eq.

Here we see superposition at work again.

We generalize this by saying that if $L[y] = g(x)$ and if $L[y_{p1}] = g_1$ and $L[y_{p2}] = g_2$, then any solution to $L[y](x) = c_1g_1 + c_2g_2$ can be expressed as

$$y(x) = c_1y_{p1}(x) + c_2y_{p2}(x) + C_1y_1(x) + \ldots + C_ny_n(x).$$

6.1

$(0, \infty)$

expl 5: Using the Wronskian, verify that the given functions form a fundamental solution set for the diff. eq. and then find a general solution.

$x^3y''' - 3x^2y'' + 6xy' - 6y = 0$, $x > 0$, $\{x, x^2, x^3\}$ ← $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$

$$W[x, x^2, x^3] = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

So, $W[x, x^2, x^3] = 2x^3$ is never zero on $(0, \infty)$. (Statement vi on pg 5) – So $\{y_1, y_2, y_3\}$ is a fund. soln set. ✓

$\boxed{\text{General soln: } y = C_1x + C_2x^2 + C_3x^3}$ (Thm 2)

expl 6: A particular solution and a fundamental solution set are given for a nonhomogeneous equation and its corresponding homogeneous equation.

a.) Find a general solution to the nonhomogeneous equation.

b.) Find the solution that satisfies the initial conditions.

$x^3y''' + xy' - y = 3 - \ln x$, $x > 0$;

$y(1) = 3$, $y'(1) = 3$, $y''(1) = 0$;

$y_p = \ln x$; $\{x,\ x\ln x,\ x(\ln x)^2\}$

(a) $y_g = \ln x + C_1x + C_2x\ln x + C_3 \cdot x(\ln x)^2$

(b) $y_g' = \frac{1}{x} + C_1 + C_2\left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) + C_3\left(1 \cdot (\ln x)^2 + x \cdot 2(\ln x)\frac{1}{x}\right)$

$y_g' = \frac{1}{x} + C_1 + C_2(\ln x + 1) + C_3((\ln x)^2 + 2 \cdot \ln x)$

$y_g'' = -x^{-2} + C_2\left(\frac{1}{x}\right) + C_3\left(2 \cdot \ln x \cdot \frac{1}{x} + 2\left(\frac{1}{x}\right)\right)$

$y(1) = 3 \rightarrow 3 = \cancel{\ln 1}^{0} + C_1(1) + \cancel{C_2 \cdot 1 \cdot \ln 1}^{0} + \cancel{C_3 \cdot 1 (\ln 1)^2}^{0}$

$3 = C_1$

$y'(1) = 3 \rightarrow 3 = \frac{1}{1} + C_1 + C_2(\cancel{\ln 1}^{0} + 1) + C_3((\cancel{\ln 1})^{2\,0} + \cancel{2 \ln 1}^{0})^{0}$

$3 = 1 + C_1 + C_2$

$3 = 1 + 3 + C_2 \rightarrow C_2 = -1$

$y''(1) = 0$

(Continuation of #6, pg 6)

$$0 = -\frac{1}{1^2} + c_2\left(\frac{1}{1}\right) + c_3\left(\cancel{2\cdot\ln 1}\cdot\frac{1}{1} + 2\left(\frac{1}{1}\right)\right)$$

$$0 = -1 + c_2 + 2c_3 \quad (\text{but } c_2 = -1)$$

$$0 = -1 + -1 + 2c_3$$

$$2 = 2c_3$$

$$c_3 = 1$$

Soln.

$$\boxed{y = \ln x + 3\cdot x - x\ln x + x(\ln x)^2}$$

expl 7: Let $L[y]=y'''-xy''+4y'-3xy$, $y_1(x)=\cos 2x$, *and* $y_2(x)=-1/3$. Show that $L[y_1]=x\cos 2x$ *and* $L[y_2]=x$. Then use the superposition principle to find the solution to the diff. eq. $L[y]=7x\cos 2x-3x$.

First, show that y_1 is a solution to the diff. eq. $y'''-xy''+4y'-3xy=x\cos 2x$.

Show $L[y_1]=x\cos 2x$

(Show $y_1=\cos 2x$ is a soln to

$y'''-xy''+4y'-3xy=x\cos 2x$

We know $y_1=\cos 2x$, $y_1'=-2\sin 2x$, $y_1''=-4\cos 2x$, $y_1'''=8\sin 2x$

Check: $y'''-xy''+4y'-3xy$

$\cancel{8\sin 2x}+4x\cos 2x\cancel{-8\sin 2x}-3x\cos 2x \stackrel{?}{=} x\cos 2x$

$x\cos 2x \stackrel{?}{=} x\cos 2x$ ✓

$\therefore L[y_1]=x\cos 2x$

Show $L[y_2]=x$

(Show $y_2=\frac{-1}{3}$ is a soln to $y'''-xy''+4y'-3xy=x$)

We know $y_2=\frac{-1}{3}$, $y_2'=0$, $y_2''=0$, $y_2'''=0$

Check: $y'''-xy''+4y'-3xy$

$0-0+0-3x\left(\frac{-1}{3}\right)\stackrel{?}{=}x$

$x\stackrel{?}{=}x$ ✓

$\therefore L[y_2]=x$.

Soln to $L[y]=7x\cos 2x-3x$ is

$y=7\cdot y_1-3y_2$

$y=7\cos 2x-3\left(\frac{-1}{3}\right) \Rightarrow \boxed{y=7\cos 2x+1}$