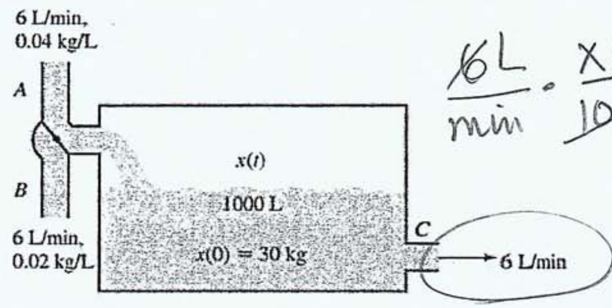
We have studied how to solve differential equations in so many different ways. But what if there was an easier way? One that would use only algebra?

Differential Equations
Class Notes
Laplace Transforms for Differential Equations (Sections 7.1 and 7.2)

Consider the tank with valved input feeders shown here. At time $t = 0$, valve A is opened, letting in a brine solution (at a concentration of 0.04 kg/L) that flows at a constant rate of 6 L/minute. At $t = 10$ minutes, valve A is closed and B is opened, letting in a brine solution (at a concentration of 0.02 kg/L) that flows at a constant rate of 6 L/minute.



$\frac{6L}{min} \cdot \frac{x(t)}{1000L} = \frac{3x}{500}$

Initially, 30 kg of salt is dissolved in the tank which has a volume of 1000 L. The outlet pipe C, which empties the tank at a constant rate of 6 L/minute, maintains the contents of the tank at constant volume. Assuming the tank is kept well stirred, determine the amount of salt in the tank at time $t > 0$.

x = amt of salt after t minutes.

As before, we know that $\frac{dx}{dt} = \textit{input rate} - \textit{output rate}$. But how do we define the input rate? It would be

We have a piecewise function for the input rate.

$$g(t) = \begin{cases} 0.04 \text{ kg/L} \times 6 \text{ L/min} = 0.24 \text{ kg/min}, & 0 < t < 10 \text{ (valve } A), \\ 0.02 \text{ kg/L} \times 6 \text{ L/min} = 0.12 \text{ kg/min}, & t > 10 \text{ (valve } B). \end{cases}$$

$x(0) = 30$

Hence, our problem is $\frac{dx}{dt} + \frac{3}{500}x = g(t)$ with initial value $t(0) = 30$.

To solve this as we have done before, we would need to break up the time interval $(0, \infty)$ into the two intervals $(0, 10)$ and $(10, \infty)$. Once we did that, the diff. eq. would be pretty straightforward. However, in the graph of $g(t)$, there is a **jump discontinuity** that would require a bit of maneuvering to get past.

But, is there an easier way? We will study Laplace transforms as an alternative. It is more convenient to solve initial value problems for linear, constant-coefficient equations this way when the forcing term contains jump discontinuities. First, we define that the **Laplace transform** (Pierre Laplace, 1779) of a function $f(t)$, defined on $[0, \infty)$, is given by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

This transforms a function in t into a function in s.

A **forcing term** (or forcing function) in a diff. eq. is a function that depends solely on time.

An alternative symbol is $\mathcal{L}$ (a cursive capital L).

$F(s) = \mathcal{L}\{f(t)\}$

We will look into this a lot more. For now, know that we are exchanging a linear, constant-coefficient differential equation in the t-domain for a simpler algebraic equation in the s-domain.

The book continues the discussion of this tank if you are interested. (Section 7.1)

We will let stand the definition given on page 1 but will also clarify that the Laplace transform takes a function $f(t)$, defined on $[0, \infty)$, and outputs a function F defined as on page 1. The **domain** of $F(s)$ is all the values of s for which the integral exists. The Laplace transform of the function $f(t)$ is **denoted** by F or $\mathcal{L}\{f\}$.

This integral is an *improper* integral. More precisely, $F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{N\to\infty} \int_0^N e^{-st} f(t)dt$

whenever the limit exists. We will pick values for s so that these limits do exist.

expl 1: Use the definition of the Laplace transform to determine it for $f(t) = te^{3t}$.

$$F(s) = \int_0^\infty e^{-st} \cdot te^{3t}\, dt$$

$$= \int_0^\infty t\, e^{(-s+3)t}\, dt$$

$$= \left(\frac{t}{3-s} - \frac{1}{(3-s)^2}\right)e^{(3-s)t}\Big]_0^\infty$$

$$= \lim_{N\to\infty}\left[\left(\frac{t}{3-s} - \frac{1}{(3-s)^2}\right)e^{(3-s)t}\right]_0^N$$

$$= \lim_{N\to\infty}\left[\left(\frac{N}{3-s} - \frac{1}{(3-s)^2}\right)e^{(3-s)N}\right] - \left(\frac{0}{(3-s)} - \frac{1}{(3-s)^2}\right)\cancel{e^{(3-s)0}}\, 1$$

Put it in the integral and simplify. You will need an integral formula.

Consider when $y = e^t$ diverges and when it does *not*.

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$$\int x e^{ax}\, dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$$

$a = -s+3$

$a = 3-s$

If $3-s<0$

$-s<-3$

$s>3$, then $e^{(3-s)N} \to 0$ as $N\to\infty$.

(assuming $s>3$)

$$= 0 + \frac{1}{(3-s)^2} = \boxed{\frac{1}{(3-s)^2} = F(s)}$$

or $\mathcal{L}\{te^{3t}\}$

Constrain s to those values where the limit is finite.

(Book has $F(s) = \frac{1}{(s-3)^2}$

7.1/7.2

expl 2: Use the definition of the Laplace transform to determine it for this piecewise function.

$$f(t)=\begin{cases} e^{2t}, & 0<t<3 \\ 1, & 3<t \end{cases}$$

Form two integrals and simplify.

$$F(s)=\int_0^3 e^{-st}\cdot e^{2t}\,dt+\int_3^\infty e^{-st}\cdot 1\,dt$$

Constrain s to those values where the limit is finite and the output is defined.

$$=\int_0^3 e^{(-s+2)t}\,dt+1\int_3^\infty e^{-st}\,dt$$

$u=(2-s)t$ $\quad$ $u=-st$

$du=(2-s)dt$ $\quad$ $du=-s\,dt$

$\frac{1}{2-s}du=dt$ $\quad$ $-\frac{1}{s}du=dt$

$$=\frac{1}{2-s}\int_0^3 e^u\,du-\frac{1}{s}\int_3^\infty e^u\,du$$

$$=\frac{1}{2-s}e^{(2-s)t}\Big]_0^3-\frac{1}{s}e^{-st}\Big]_3^\infty$$

$$=\frac{1}{2-s}e^{(2-s)\cdot 3}-\frac{1}{2-s}\cancel{e^{(2-s)0}}^{\,1}-\frac{1}{s}\lim_{N\to\infty}e^{-st}\Big]_3^N$$

The output is *not* defined when $s=2$. Deal with it separately.

$$=\frac{1}{2-s}\left(e^{6-3s}-1\right)-\frac{1}{s}\left[\lim_{N\to\infty}e^{-s(N)}-e^{-s(3)}\right]$$

If $s>0$, $e^{-Ns}\to 0$ as $N\to\infty$.

$$=\frac{e^{6-3s}-1}{2-s}+\frac{e^{-3s}}{s}=F(s)$$ but only defined for positive $s\neq 2$

For $s=2$, reevaluate $\int_0^3 e^{-2t}e^{2t}\,dt+\int_3^\infty e^{-2t}\cdot 1\,dt$

$$=\int_0^3 1\,dt+\int_3^\infty e^{-2t}\,dt$$

$u=-2t$

$du=-2dt$

$-\frac{1}{2}du=dt$

$$=t\Big]_0^3-\frac{1}{2}\int_{-6}^\infty e^u\,dt$$

$$=3-0-\frac{1}{2}e^{-2t}\Big]_3^\infty=3-\frac{1}{2}\lim_{N\to\infty}e^{-2N}+\frac{1}{2}e^{-6}$$ ($\lim_{N\to\infty}e^{-2N}\to 0$)

end of #2
(7.1/7.2 notes)

For $s=2$,

$$\boxed{F(s) = 3 + \frac{1}{2}e^{-6}}$$

For all other <u>positive</u> s,

$$\boxed{F(s) = \frac{e^{6-3s} - 1}{2-s} + \frac{e^{-3s}}{s}}$$

Laplace Transforms Tables:

Luckily, we are *not* the first to travel this road. Here is a table of Laplace transforms for common functions. Notice the constraints put on s.

TABLE 7.1 Brief Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\dfrac{1}{s}, \quad s > 0$
e^{at}	$\dfrac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\dfrac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\dfrac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$

Linearity of Laplace Transforms:

We also have this notion that will help us break up more complicated functions and deal with their terms individually.

Linearity of the Transform

Theorem 1. Let f, f_1, and f_2 be functions whose Laplace transforms exist for $s > \alpha$ and let c be a constant. Then, for $s > \alpha$,

(2) $$\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\} ,$$

(3) $$\mathcal{L}\{cf\} = c\mathcal{L}\{f\} .$$

$\mathcal{L}\{f\}$

expl 3: Use the Laplace transform table and the linearity of the Laplace transform to determine the following.

$\mathcal{L}\left\{e^{3t}\sin 6t - t^3 + e^t\right\}$

$= \mathcal{L}\{e^{3t}\sin 6t\} - \mathcal{L}\{t^3\} + \mathcal{L}\{e^t\}$

$$F(s) = \frac{6}{(s-3)^2+6^2} - \frac{3!}{s^4} + \frac{1}{s-1}$$

$s>3 \qquad s>0 \qquad s>1$

Combine the constraints for s into a single constraint.

Definition: Jump discontinuity: A function $f(t)$ is said to have a **jump discontinuity** at $t_0 \in (a,b)$ if $f(t)$ is discontinuous at t_0 but the one-sided limits $\lim_{t\to t_0^-} f(t)$ and $\lim_{t\to t_0^+} f(t)$ exist as finite numbers.

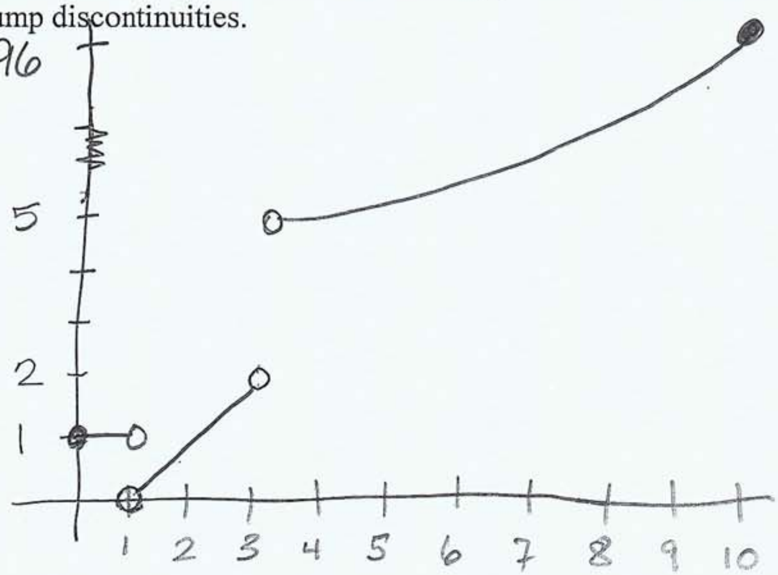
The input function $g(t)$ for the mixing tank problem has a jump discontinuity at $t_0 = 10$.

Definition: Piecewise continuous: A function $f(t)$ is said to be **piecewise continuous on a finite interval [a, b]** if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(t)$ is said to be **piecewise continuous on [0, ∞)** if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

expl 4: Sketch the graph to determine whether the function is continuous, piecewise continuous, or neither on [0, 10]. Denote any jump discontinuities.

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t-1, & 1 < t < 3 \\ t^2-4, & 3 < t \le 10 \end{cases}$$



Piecewise conti on $[0,10]$.

Jump discontinuities at $t = 1, 3$.

Definition: Exponential Order: A function $f(t)$ is said to be of **exponential order α** if there exist positive constants T and M such that $|f(t)| \le Me^{\alpha t}$ for all $t \ge T$.

For example, $f(t) = e^{3t}\sin 6t$ is of exponential order $\alpha = 3$ since $|e^{3t}\sin 6t| \le e^{3t}$. (Here, $M = 1$ and T is any positive constant.)

A common way to determine if a function $f(t)$ is of exponential order α is to consider the limit $\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}}$ and try to show it is a constant (really, 0).

We can use **L'Hôpital's Rule** (below) to find such limits as needed.

L'Hospital's/L'Hôpital's Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ then,

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, a is a number, ∞ or $-\infty$

Source: https://tutorial.math.lamar.edu/pdf/calculus_cheat_sheet_limits.pdf

Handout: Calculus Cheat Sheet: (Paul Dawkins):
The above L'Hôpital's Rule and much more about limits is available on
https://tutorial.math.lamar.edu/pdf/calculus_cheat_sheet_limits.pdf

expl 5: Is the function below of exponential order? If so, what value of α would you assign it?

$f(t) = 100e^{49t}$

Form the $\lim_{t\to\infty} \frac{f(t)}{e^{\_\,t}}$ leaving α blank. What would α have to be so that this limit was 0?

$$\lim_{t\to\infty} \frac{100e^{49t}}{e^{50t}} = \lim_{t\to\infty} \frac{100}{e^{t}} = 0$$

So, $f(t)$ is of exponential order 50.

expl 6: Is the function below of exponential order? If so, what value of α would you assign it?

$f(t) = t^2 + 2$

$$\lim_{t\to\infty} \frac{t^2+2}{e^t} = \frac{\infty}{\infty}$$

Let α be 1 and form the $\lim_{t\to\infty} \frac{f(t)}{e^t}$. This will be indeterminate (∞/∞) so use L'Hôpital's Rule (twice).

L'Hopitals Rule says

$$\lim_{t\to\infty} \frac{t^2+2}{e^t} = \lim_{t\to\infty} \frac{2t}{e^t} = \frac{\infty}{\infty}.$$

Again, L'Hopital's Rule

$$= \lim_{t\to\infty} \frac{2}{e^t} = \frac{2}{\infty} = 0$$

So, $f(t) = t^2 + 2$ is of exponential order 1.

Theorem: Conditions for the Existence of the Laplace Transform:

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$. (Proof shown in book.)