We are given some function in $s$ (a Laplace transform) and asked for the function in $t$ from which it came.

Differential Equations
Class Notes
Inverse Laplace Transforms (Section 7.4)

Since the Laplace transform is a function (or mapping) that takes an input function $f(t)$ and outputs its Laplace transform $\mathcal{L}\{f(t)\}$ or $F(s)$, we should be able to undo that with the **inverse mapping**. And we shall!

We may be given a Laplace transform whose form we have on the brief table of Laplace transforms given in the first section of this chapter. However, when it is not that simple, we may have to rewrite the function $F(s)$ by using partial fraction decomposition. Other tricks of the trade for rewriting this function $F(s)$ will be shown too.

Let's begin with a formal definition of the Inverse Laplace Transform and its linearity.

**Inverse Laplace Transform**

**Definition 4.** Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies

$$\mathcal{L}\{f\} = F, \tag{2}$$

then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $f = \mathcal{L}^{-1}\{F\}$.

**Linearity of the Inverse Transform**

**Theorem 7.** Assume that $\mathcal{L}^{-1}\{F\}$, $\mathcal{L}^{-1}\{F_1\}$, and $\mathcal{L}^{-1}\{F_2\}$ exist and are continuous on $[0, \infty)$ and let $c$ be any constant. Then

$$\mathcal{L}^{-1}\{F_1 + F_2\} = \mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\}, \tag{3}$$

$$\mathcal{L}^{-1}\{cF\} = c\mathcal{L}^{-1}\{F\}. \tag{4}$$

When solving a diff. eq., we want a function that is continuous. Fortunately, it can be shown that if two different functions have the same Laplace transform, at most one of them can be continuous. That continuous one is the solution we seek.

Again, we will want that table of common Laplace transforms in hand as we complete these problems.

Use the notation $\mathcal{L}^{-1}\{F\}$ as you work.

expl 1: Determine the inverse Laplace transform of the given function.

$$\mathcal{L}\{f\} = \frac{6}{(s-1)^4}$$

Compare this form to those in the Laplace transforms table.

$$\mathcal{L}^{-1}\mathcal{L}\{f\} = \mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\}$$

$a=1$

$n=3$

$$f(t) = e^{t}t^{3}$$

expl 2: Determine the inverse Laplace transform of the given function.

$$\mathcal{L}\{f\} = \frac{4}{s^2+9}$$

Start on the bottom and find an appropriate Laplace transform. Then deal with the top so it works out too.

$$\mathcal{L}^{-1}\mathcal{L}\{f\} = \mathcal{L}^{-1}\left\{\frac{4}{s^2+9}\right\}$$

$$f = \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$b=3$

$$\boxed{f(t) = \frac{4}{3}\sin 3t}$$

As we said, sometimes the form $F(s)$ is given so that the function $f(t)$ is *not* readily deciphered. We might need partial fraction decomposition or completing the square to rewrite the given $F(s)$. A brief review of both procedures follows.

**Recall: Completing the Square:**

Completing the square is a technique that forces an expression in the form $as^2 + bs + c$ into the form $a(s-h)^2 + k$.

We will use completing the square to rewrite an expression such as $s^2 - 6s + 13$ as $(s-3)^2 + 2^2$ which is more accessible for the Laplace Transform Table.

$\{(s-a)^2 + b^2\}$

**To recall the procedure, look at this example:**

$(s-3)^2 = s^2 - 6s + 9$

$\frac{-6}{2} = -3 \quad (-3)^2 = 9$

What is the relationship between the coefficient of the $s$-term and the constant at the end?

So, we are interested in going from $s^2 - 6s + 9$ back to the $(s-3)^2$ form. If we were just given the $s^2 - 6s$ part, how would we figure out the constant that "completed" $s^2 - 6s$ so that we could factor it as $(s-3)^2$?

And what if we were given $s^2 - 6s + 34$? Complete the square here.

$$s^2 - 6s + 9 + 34 - 9 = (s-3)^2 + 25$$

$\frac{-6}{2} = -3 \quad (-3) = 9$

expl 3: Determine the inverse Laplace transform of the given function.

$$\mathcal{L}\{f\} = \frac{5}{s^2 - 6s + 34}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{5}{(s-3)^2 + 25}\right\}$$

$a = 3$

$b = 5$

$$f(t) = e^{3t} \cdot \sin 5t$$

$$\underline{7.4} \qquad \mathcal{L}^{-1}\mathcal{L}\{f\} = f$$

expl 4: Determine the inverse Laplace transform of the given function.

$$\mathcal{L}\{f\} = \frac{3s-15}{2s^2-4s+10}$$

Remember if the $s^2$ term has a coefficient other than 1, we need to factor it out before completing the square.

$$f(t) = \mathcal{L}^{-1}\left\{\frac{3s-15}{2(s-1)^2+8}\right\}$$

$a=1$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{3s-15}{(s-1)^2+4}\right\}$$

$a=1$
$b=2$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{3(s-1)-12}{(s-1)^2+4}\right\}$$

$$= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} - \frac{6}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\}$$

$$\boxed{f(t) = \frac{3}{2}e^{1t}\cos 2t - 3e^{1t}\sin 2t}$$

Completing the square

$$2s^2-4s+10$$

$$= 2(s^2-2s+1)+10-2$$

$$\frac{-2}{2} = -1$$

$$(-1)^2 = 1$$

$$= 2(s-1)^2+8$$

**Recall: Partial Fraction Decomposition:**

We briefly review this method. Recall from calculus that a rational function of the form $P(s)/Q(s)$, where $P(s)$ and $Q(s)$ are polynomials with the degree of $P$ less than the degree of $Q$, has a partial fraction expansion whose form is based on the linear and quadratic factors of $Q(s)$. (We assume the coefficients of the polynomials to be real numbers.) There are three cases to consider:

1. Nonrepeated linear factors.
2. Repeated linear factors.
3. Quadratic factors.

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## 1. Nonrepeated Linear Factors

If $Q(s)$ can be factored into a product of distinct linear factors,

$$Q(s) = (s - r_1)(s - r_2) \cdots (s - r_n) ,$$

where the $r_i$'s are all distinct real numbers, then the partial fraction expansion has the form

$$\boxed{\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \frac{A_2}{s - r_2} + \cdots + \frac{A_n}{s - r_n} ,}$$

where the $A_i$'s are real numbers. There are various ways of determining the constants $A_1, \ldots, A_n$. In the next example, we demonstrate two such methods.

## 2. Repeated Linear Factors

If $s - r$ is a factor of $Q(s)$ and $(s - r)^m$ is the highest power of $s - r$ that divides $Q(s)$, then the portion of the partial fraction expansion of $P(s)/Q(s)$ that corresponds to the term $(s - r)^m$ is

$$\boxed{\frac{A_1}{s - r} + \frac{A_2}{(s - r)^2} + \cdots + \frac{A_m}{(s - r)^m} ,}$$

where the $A_i$'s are real numbers.

## 3. Quadratic Factors

If $(s - \alpha)^2 + \beta^2$ is a quadratic factor of $Q(s)$ that cannot be reduced to linear factors with real coefficients and $m$ is the highest power of $(s - \alpha)^2 + \beta^2$ that divides $Q(s)$, then the portion of the partial fraction expansion that corresponds to $(s - \alpha)^2 + \beta^2$ is

$$\frac{C_1 s + D_1}{(s - \alpha)^2 + \beta^2} + \frac{C_2 s + D_2}{[(s - \alpha)^2 + \beta^2]^2} + \cdots + \frac{C_m s + D_m}{[(s - \alpha)^2 + \beta^2]^m} .$$

As we saw in Example 4, page 369, it is more convenient to express $C_i s + D_i$ in the form $A_i(s - \alpha) + \beta B_i$ when we look up the Laplace transforms. So let's agree to write this portion of the partial fraction expansion in the equivalent form

$$\boxed{\frac{A_1(s - \alpha) + \beta B_1}{(s - \alpha)^2 + \beta^2} + \frac{A_2(s - \alpha) + \beta B_2}{[(s - \alpha)^2 + \beta^2]^2} + \cdots + \frac{A_m(s - \alpha) + \beta B_m}{[(s - \alpha)^2 + \beta^2]^m} .}$$

$$F(s) = \mathcal{L}\{f(t)\}$$

expl 5: Determine the inverse Laplace transform of the given function.

$$F(s) = \frac{5s^2+34s+53}{(s+3)^2(s+1)}$$

Set up the partial fraction decomposition with the appropriate terms and solve for the constants.

$$\frac{5s^2+34s+53}{(s+3)^2(s+1)} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s+1}$$

(multiply all by $(s+3)^2(s+1)$)

$$5s^2+34s+53 = A(s+3)(s+1) + B(s+1) + C(s+3)^2$$

$$5s^2+34s+53 = A(s^2+4s+3) + B(s+1) + C(s^2+6s+9)$$

$$\begin{aligned} 5 &= A + C \\ 34 &= 4A + B + 6C \\ 53 &= 3A + B + 9C \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 4 & 1 & 6 & 34 \\ 3 & 1 & 9 & 53 \end{array}\right] \rightarrow \begin{aligned} A &= -1 \\ B &= 2 \\ C &= 6 \end{aligned}$$

$$F(s) = \frac{-1}{s+3} + \frac{2}{(s+3)^2} + \frac{6}{s+1}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{s+1}\right\}$$

$n=1$

$$f(t) = -1\,\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} + 6\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$f(t) = -e^{-3t} + 2 \cdot e^{-3t} \cdot t^1 + 6e^{-t}$$

$$\boxed{f(t) = -e^{-3t} + 2te^{-3t} + 6e^{-t}}$$

expl 6: Determine the inverse Laplace transform of the given function.

$$F(s)=\frac{7s^2-41s+84}{(s^2-4s+13)(s-1)}$$

Use completing the square to rewrite the irreducible quadratic factor before setting up the partial fraction decomposition.

$$\frac{7s^2-41s+84}{(s^2-4s+13)(s-1)} = \frac{A}{s-1} + \frac{B(s-2)+3C}{(s-2)^2+3^2}$$

$\alpha = 2$, $\beta = 3$

(mult all be $(s^2-4s+13)(s-1)$)

$$s^2-4s+13$$
$$=s^2-4s+4+13-4$$
$$\frac{-4}{2}=-2 \qquad (-2)^2=4$$
$$=(s-2)^2+3^2$$

$$7s^2-41s+84 = A(s^2-4s+13)+(B(s-2)+3C)(s-1)$$

$$7s^2-41s+84 = A(s^2-4s+13)+B(s^2-3s+2) + C(3s-3)$$

$$7 = A + B$$
$$-41 = -4A - 3B + 3C$$
$$84 = 13A + 2B - 3C$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ -4 & -3 & 3 & -41 \\ 13 & 2 & -3 & 84 \end{array}\right] \rightarrow \begin{array}{l} A=5 \\ B=2 \\ C=-5 \end{array}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{s-1} + \frac{2(s-2)-15}{(s-2)^2+3^2}\right\}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{5}{s-1}\right\} + 2\cdot\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+3^2}\right\} - 5\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+3^2}\right\}$$

$$f(t) = 5\cdot e^{1t} + 2\cdot e^{2t}\cos 3t - 5\cdot e^{2t}\sin 3t$$