A few choice properties of Laplace transforms will help us solve these initial value problems. Can you guess them?

Differential Equations
Class Notes
Solving Initial Value Problems with Laplace Transforms (Section 7.5)

We have solved these equations before. However, we were required to find a general solution *first* and then use the initial values to narrow down that solution. With Laplace transforms, we do *not* need to obtain the general solution first. How would we do that?

We will take the Laplace transform of both sides of our diff. eq.. Do we have any Laplace properties that involve the initial values that can be put to use. Yes we do!

If we are given a diff. eq. and a few initial values (*always* for $t = 0$ or we would need to use a translation to make that so), we can use these properties (from section 7.3) copied below.

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$
$$\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$$
$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Method for Solving Differential Equations with Laplace Transforms:

1. Take the Laplace transform of both sides of the equation.

2. Use the above properties and the initial values given to you to obtain an equation in which you can isolate $\mathcal{L}\{y\}$ (also known as $Y(s)$ or $\mathcal{L}\{f\}$ in the above properties). F(S)

3. Determine the inverse Laplace transform of the solution by looking it up in a table. You may need partial fraction decomposition or some other device to get there.

4. Dance like no one is watching.

We will tacitly assume the solution is a piecewise continuous function on $[0, \infty)$ and of exponential order. Once we have our solution, we can certainly verify those assumptions.

Not mentioned above is the linearity of Laplace transforms which we use liberally.

expl 1: Solve the initial value problem using the method of Laplace transforms.

$y'' + 6y' + 5y = 12e^t \qquad y(0) = -1, \quad y'(0) = 7$

Apply the Laplace transform to all terms. Use the Derivative Properties from the first page.

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{12e^t\}$$

$$s^2\mathcal{L}\{y\} + s - 7 + 6\left(s\mathcal{L}\{y\} + 1\right) + 5\mathcal{L}\{y\} = 12/(s-1)$$

$$s^2\mathcal{L}\{y\} + s - 7 + 6s\mathcal{L}\{y\} + 6 + 5\mathcal{L}\{y\} = 12/(s-1)$$

$$\mathcal{L}\{y\}\left(s^2+6s+5\right) + s - 1 = \frac{12}{s-1}$$

$$\mathcal{L}\{y\}\left(s^2+6s+5\right) = \frac{12}{s-1} - s + 1$$

$$= \frac{12}{s-1} - \frac{s(s-1)}{s-1} + \frac{1(s-1)}{s-1}$$

$$= \frac{12 - s^2 + s + s - 1}{(s-1)}$$

$$\mathcal{L}\{y\} = \frac{-s^2+2s+11}{(s-1)(s+1)(s+5)}$$

Once you isolate $\mathcal{L}\{y\}$, you will need partial fraction decomposition.

$$\frac{-s^2+2s+11}{(s-1)(s+1)(s+5)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+5}$$

$$-s^2+2s+11 = A\left(s^2+6s+5\right) + B\left(s^2+4s-5\right) + C\left(s^2-1\right)$$

$-1 = A + B + C$

$2 = 6A + 4B$ $\longrightarrow$ calculator matrix

$11 = 5A - 5B - C$

$A = 1$

$B = -1$

$C = -1$

(extra room to work) $\mathcal{L}\{y\} = \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{s+5}$

Get out the table of Laplace transforms.

$$\mathcal{L}^{-1}\mathcal{L}\{y\} = \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$\boxed{y = e^{1t} - e^{-1t} - e^{-5t}}$$

expl 2: Solve the initial value problem using the method of Laplace transforms.

$y'' + 4y = 4t^2 - 4t + 10 \quad y(0) = 0, \quad y'(0) = 3$

… and if your friends don't dance, then they're … no friends of mine…

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{4t^2 - 4t + 10\}$$

$$s^2\mathcal{L}\{y\} - 0 - 3 + 4\mathcal{L}\{y\}$$

$$\mathcal{L}\{y\}(s^2+4) - 3 = 4\mathcal{L}\{t^2\} - 4\mathcal{L}\{t\} + 10\mathcal{L}\{1\}$$

$n=2$, $a=0$ $\qquad n=1$

$$\mathcal{L}\{y\}(s^2+4) - 3 = 4 \cdot \frac{2!}{(s)^3} - 4 \cdot \frac{1}{s^2} + 10 \cdot \frac{1}{s}$$

$$\mathcal{L}\{y\}(s^2+4) = \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s} + 3$$

$$\mathcal{L}\{y\} = \frac{8 - 4s + 10s^2 + 3s^3}{s^3(s^2+4)}$$

$$= \frac{3s^3 + 10s^2 - 4s + 8}{s^3(s^2+4)}$$

(extra room to work)

$$\frac{3s^3+10s^2-4s+8}{s^3(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D(s)+2E}{(s-0)^2+2^2}$$

$a=0$
$b=2$

$$3s^3+10s^2-4s+8 = A(s^2(s^2+4)) + B(s(s^2+4)) + C(s^2+4) + (Ds+2E)(s^3)$$

$$3s^3+10s^2-4s+8 = A(s^4+4s^2) + B(s^3+4s) + C(s^2+4) + D(s^4) + E(2s^3)$$

$0 = A + D$
$3 = B + 2E$
$10 = 4A + C$
$-4 = 4B$
$8 = 4C$

$\rightarrow$ $B=-1$, $C=2$

$10 = 4A+2$
$8 = 4A$
$2 = A$

$3 = B+2E$
$3 = -1+2E$
$4 = 2E$
$2 = E$

$0 = A+D$
$0 = 2+D$
$-2 = D$

$$\mathcal{L}^{-1}\mathcal{L}\{y\} = \mathcal{L}^{-1}\left(\frac{2}{s} - \frac{1}{s^2} + \frac{2}{s^3} + \frac{-2s+4}{s^2+4}\right)$$

$$= 2 - t + t^2 - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 2\mathcal{L}^{-1}\frac{2}{s^2+4}$$

$$\boxed{y = 2 - t + t^2 - 2\cdot\cos 2t + 2\cdot\sin 2t}$$

Some homework problems only want you to go as far as $Y(s)$ or $\mathcal{L}\{y\}$. You do *not* need to solve for the function y.

Differential Equations Involving Piecewise Functions:
To deal with these, we will use the definition of a Laplace transform involving an integral which is this bad boy.

$$F(s)=\int_0^{\infty} e^{-st} f(t)\,dt$$

> We will take the Laplace transform of a piecewise function. Recall that involves breaking up the integral.

expl 3: Solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to this initial value problem.

$y''-y=g(t) \qquad y(0)=1, \quad y'(0)=2 \longrightarrow \mathcal{L}\{y''\}-\mathcal{L}\{y\}=\mathcal{L}\{g(t)\}$

$$\text{where } g(t)=\begin{cases}1, & t<3\\ t, & t>3\end{cases}$$

$$\mathcal{L}\{g(t)\} = \int_0^3 e^{-st}\,dt + \int_3^{\infty} e^{-st}\cdot t\,dt$$

$u=-st$

$du=-s\,dt$

$-\frac{1}{s}du=dt$

#53 $\int x e^{ax}dx=\left(\frac{x}{a}-\frac{1}{a^2}\right)e^{ax}$

$a=-s$

$$=-\frac{1}{s}\int e^{u}\,du \qquad + \lim_{N\to\infty}\left(\frac{t}{-s}-\frac{1}{s^2}\right)e^{-st}\Big]_3^N$$

$$=-\frac{1}{s}e^{-st}\Big]_0^3 \quad + \overset{\lim_{N\to\infty}}{0} - \left(\frac{3}{-s}-\frac{1}{s^2}\right)e^{-3s}$$

$$=-\frac{1}{s}\left(e^{-3s}-1\right)+\left(\frac{3}{s}-\frac{1}{s^2}\right)e^{-3s}$$

$$=\frac{-e^{-3s}}{s}+\frac{1}{s}+\frac{3e^{-3s}}{s}-\frac{e^{-3s}}{s^2}$$

$$=\frac{2e^{-3s}+1}{s}-\frac{e^{-3s}}{s^2}$$

$$\mathcal{L}\{g(t)\}=\frac{2se^{-3s}+s-e^{-3s}}{s^2}$$

(this is right side of eqn.) $\longrightarrow$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \frac{2se^{-3s} + s + e^{-3s}}{s^2}$$

$$s^2\mathcal{L}\{y\} - s - 2 - \mathcal{L}\{y\} = \ldots$$

(extra room to work)

$$\mathcal{L}\{y\}(s^2-1) - s - 2 = \frac{2se^{-3s} + s + e^{-3s}}{s^2}$$

$$(s^2-1)\mathcal{L}\{y\} = \frac{2se^{-3s} + s + e^{-3s}}{s^2} + \frac{(s+2)s^2}{s^2}$$

$$\mathcal{L}\{y\} = \frac{2se^{-3s} + e^{-3s} + s + s^3 + 2s^2}{s^2(s^2-1)}$$

Solving Higher-Order Differential Equations with Laplace:

You will be asked to solve a third-order diff. eq.. It is done in the same manner as we have seen. You may need polynomial long division (to factor a cubic expression) and partial fraction decomposition. Have fun. Dance much.