We will study the diff. eqs. concerning two simple electrical circuits. It's really shocking material!

Differential Equations
Class Notes
Electrical Circuits (Section 3.5)

A simple electrical circuit has a voltage source (like a battery or generator), a resistor, and either a capacitor or an inductor. Let's define some terms and see how these circuits generate diff. eqs.

One of the main differences between a **capacitor** and an **inductor** is that a capacitor opposes a change in voltage while an inductor opposes a change in the current. Furthermore, the inductor stores energy in the form of a magnetic field, and the capacitor stores energy in the form of an electric field. (source: https://byjus.com/physics/difference-between-capacitor-and-inductor/)

Definition: Inductance: The property of an electric circuit by which an electromotive force (voltage) is induced in it by a variation of current either in the circuit itself or in a neighboring circuit. An **inductor** is a coil of wire wrapped around an iron core.
(source: https://www.swtc.edu/Ag_Power/electrical/terms.htm)

Definition: Voltage (volts, V): Voltage is the force that makes electrons flow. It's a difference in potential energy between two different points in a circuit.

Definition: Current (amperes or **amps, A): Current** is the *rate* of the flow of electrons.

Definition: Power (Watts, W or **w):** The **power** used in a circuit is measured in watts. **Watts** are calculated by multiplying the voltage by the current.

Definition: Resistance: This is the measure of how well something conducts electricity. If it has a low resistance, the object is a great conductor of electricity. If it has a high resistance, that means that it doesn't conduct electricity well. A **resistor** is a device that resists current flow.
(above definitions source: https://dewesoft.com/blog/volts-and-currents-explained)

Definition: Capacitance: Capacitance is the capability of a material object or device to store electric charge. It is measured by the charge in response to a difference in electric potential, expressed as the ratio of those quantities. (source: www.wikipedia.org)

TABLE 3.3 Common Units and Symbols Used With Electrical Circuits

Quantity	Letter Representation	Units	Symbol Representation
Voltage source	E	volt (V)	-o- Generator -|⊢ Battery
Resistance	R	ohm (Ω)	-ʌʌʌ-
Inductance	L	henry (H)	-ooo-
Capacitance	C	farad (F)	⊣⊢
Charge	q	coulomb (C)	
Current	I	ampere (A)	

We will use these units, symbols, and abbreviations.

The Physics Behind the Math:
The story goes that good, old Gustav Robert Kirchhoff in 1859 came up with these gems.

1. **Kirchhoff's Current Law**: The algebraic sum of the currents flowing into any junction point must be zero.

2. **Kirchhoff's Voltage Law**: The algebraic sum of the instantaneous changes in potential (voltage drops) around any closed loop must be zero.

Here are pictures of the two circuits we will be exploring.

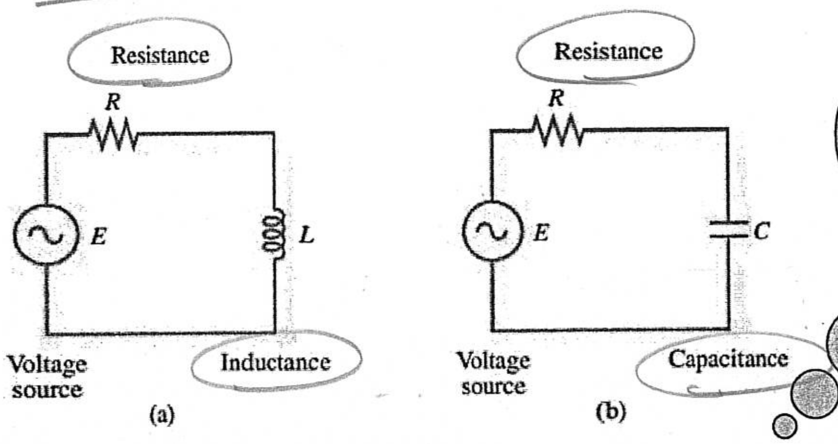


Figure 3.13 (a) *RL* circuit and (b) *RC* circuit

Kirchhoff's Current Law implies that the same current passes through all elements in each circuit.

To apply Kirchhoff's voltage law, we need to know the voltage drop across each element of the circuit. We state them here.

1. According to Ohm's Law, the voltage drop E_R across a resistor is proportional to the current I passing through the resistor. The proportionality constant R is called the **resistance.**

$$E_R = RI \text{ where } R \text{ is a real number}$$

2. Faraday's Law and Lenz's Law can be used to show that the voltage drop E_L across an inductor is proportional to the instantaneous *rate of change* of the current I. The proportionality constant L is called the **inductance.**

$$E_L = L\frac{dI}{dt} \text{ where } L \text{ is a real number}$$

3. The voltage drop E_C across a capacitor is proportional to the electrical charge q on the capacitor. The constant C is called the **capacitance.**

$$E_C = \frac{1}{C}q \text{ where } C \text{ is a real number}$$

Rationale and Method for RL Circuits:

A voltage source is assumed to add voltage (or potential energy) to a circuit. Let $E(t)$ be the voltage supplied by a circuit at time t. We now apply Kirchhoff's Voltage Law to the RL (resistance, inductance) circuit shown on the previous page. This gets us

linear eqns (2.3)

$$E_L + E_R = E(t)$$

$$L\frac{dI}{dt} + RI = E(t)$$

$$\frac{dI}{dt} + \underbrace{\frac{R}{L}}_{P(t)} I = \frac{E(t)}{L}$$

linear

This is linear, in the form $\frac{dy}{dx} + P(x)\cdot y = Q(x)$. So, we will find $\mu(x) = e^{\int P(x)dx}$.

We want to find $I(t)$, the current at time t.

In this case, our integrating factor will be

$$\mu(t) = e^{\cancel{\int P(x)dx}} = e^{\int (R/L)dt} = e^{Rt/L}.$$

$e^{\int P(t)dt}$

We know that the solution to the general linear equation is $y = \dfrac{\int \mu(x)Q(x)dx + c}{\mu(x)}$. Hence, the solution to this circuit's diff. eq. is $\boxed{I(t) = e^{-Rt/L}\left[\int e^{Rt/L} \cdot \frac{E(t)}{L} dt + K\right]}$. Here, K is the constant of integration (a real number).

We will find K if they give us an initial point.

$I(0) = 1$ $R = 5$ $L = 0.05$

expl 1: An RL circuit with a $5-\Omega$ resistor and a 0.05-H inductor carries a current of 1 A at time $t = 0$, at which time a voltage source $E(t) = 5\cos(120\,t)$ V is added. Determine the subsequent inductor current and voltage.

We want $I(t)$ and E_L. Recall, R is the resistance and L is the inductance.

$$I(t) = e^{-5t/0.05}\left[\int e^{5t/0.05} \frac{5\cos(120t)}{0.05} dt + K\right]$$

$$= e^{-100t}\left[100 \underbrace{\int e^{100t}\cos(120t)\,dt}_{} + K\right]$$

There is room to solve on the next page.

$$I(t) = e^{-100t} \cdot \left[100 \int e^{100t} \cos(120t)\, dt + k \right]$$

$$a = 120 \qquad b = 100$$

(extra room for work)

Eek! (#107)

$$\int e^{bx} \cos(ax)\, dx = \frac{1}{a^2 + b^2} e^{bx} \left(a \sin(ax) + b \cos(ax) \right)$$

Perform the integration and simplify to find $I(t)$ with a constant of integration in place. Then use the initial point given to find the constant. Do *not* forget to find the voltage which is

$$E_L = L \frac{dI}{dt}.$$

$$I(t) = e^{-100t} \left[100 \cdot \frac{1}{120^2 + 100^2} e^{100t} \left(120 \sin(120t) + 100 \cos(120t) \right) + k \right] \qquad k \in \mathbb{R}$$

$$I(t) = \frac{100}{24{,}400} \left(120 \sin(120t) + 100 \cos(120t) \right) + e^{-100t} \cdot K$$

$$I(t) = \frac{30}{61} \sin(120t) + \frac{25}{61} \cos(120t) + e^{-100t} \cdot K$$

we know $I(0) = 1$

$$1 = \frac{30}{61} \underbrace{\sin 0}_{\to 0} + \frac{25}{61} \underbrace{\cos 0}_{\to 1} + \underbrace{e^{-100(0)}}_{\to 1} \cdot K$$

$$1 - \frac{25}{61} = K \qquad \to \qquad K = \frac{36}{61}$$

$$\boxed{I(t) = \frac{30}{61} \sin(120t) + \frac{25}{61} \cos(120t) + \frac{36}{61} e^{-100t}}$$ current

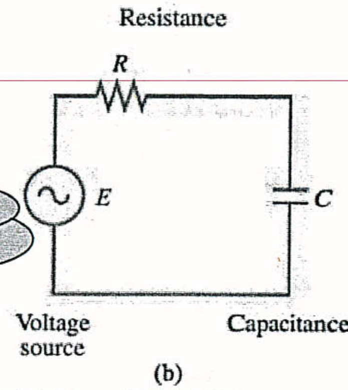
$$E_L = L \frac{dI}{dt} = (0.05) \left[\frac{30}{61}(120) \cos(120t) - \frac{25}{61}(120) \sin(120t) + \frac{36}{61}(-100) e^{-100t} \right]$$

$$\boxed{E_L = \frac{180}{61} \cos(120t) - \frac{150}{61} \sin(120t) - \frac{180}{61} e^{-100t}}$$ voltage drop

$$q(t) = e^{-t/RC}\left[\frac{1}{R}\int e^{t/RC}E(t)\,dt + k\right]$$

Rationale and Method for RC Circuits:

A voltage source is assumed to add voltage (or potential energy) to a circuit. Let $E(t)$ be the voltage supplied by a circuit at time t. We now apply Kirchhoff's Voltage Law to the RC (resistance, capacitance) circuit shown here. This gets us



$$E_R + E_C = E(t)$$

$$RI + \frac{1}{C}q = E(t)$$

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$\boxed{\frac{dq}{dt} + \frac{1}{RC}q = \frac{E(t)}{R}}$$

q: electric charge (coulombs)
I: current (amps)

Holy linearity, Batman! *That's linear too.* (Robin does not know this but it's also separable if $E(t)$ is constant.)

$$q(t) = \frac{\int \mu(t)\frac{E(t)}{R}\,dt}{\mu(t)}$$

We solve this with our method for linear equations (using $\mu(t) = e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$) to get us

$q(t) = e^{\frac{-t}{RC}}\left[\frac{1}{R}\int e^{\frac{t}{RC}} \cdot E(t)\,dt + k\right]$. Use K for the constant of integration (a real number).

expl 2: Assume $E(t)$ is constant at V volts. Solve for $q(t)$, the capacitor charge for $t > 0$. Recall, the resistance is R Ohms and the capacitance is C farads. Further, assume $q(0) = Q$ coulombs and solve for the constant of integration. I have substituted V in for $E(t)$ below.

$$q(t) = e^{\frac{-t}{RC}}\left[\frac{1}{R}\int e^{\frac{t}{RC}} \cdot V\,dt + k\right]$$

$$= e^{-t/RC}\left[\frac{V}{R}\int e^{t/RC}\,dt\right]$$

$u = t/RC$

$du = \frac{1}{RC}dt$

$RC\,du = dt$

$$q(t) = e^{-t/RC}\left[\frac{V}{\cancel{R}}\frac{\cancel{R}C}{1}\int e^{u}\,du\right]$$

$$= e^{-t/RC}\left[VC\,e^{t/RC} + K\right]$$

$$q(t) = VC + Ke^{-t/RC}$$

$q(0) = Q$

$Q = VC + K\cancel{e^{-0/RC}}$ $\to 1$

$K = Q - VC$

$$\boxed{q(t) = VC + (Q - VC)e^{-t/RC}}$$

$q(t)$ = electric charge at time t.

$E(t) = V = 5$

$Q=0 \qquad R=100 \qquad C = 10^{-12}$

expl 3: The pathway for a binary electrical signal between gates in an integrated circuit can be modeled as an RC circuit. The voltage source models the transmitting gate and the capacitor models the receiving gate. Typically, the resistance is 100 Ohms and the capacitance is very small at 10^{-12} farads. If the capacitor is initially uncharged ($Q = 0$) and the transmitting gate changes instantaneously from 0 to 5 volts, how long will it take for the voltage at the receiving gate to reach 3 volts? (This is the time it takes to transmit a logical "1".)

When is $E_c = 3$?

(derived in example 2)

$$q(t) = VC + (Q - VC)e^{-t/RC}$$

Use the formula for $q(t)$ we derived in example 2. We know that $Q = 0$, $V = 5$, $R = 100$, and $C = 10^{-12}$.

$$RC = 100\,(10^{-12}) = 10^2\,10^{-12} = 10^{-10}$$

$$q(t) = 5\times 10^{-12} + (0 - 5\times 10^{-12})e^{-10^{10}t}$$

$$= 5\times 10^{-12} - (5\times 10^{-12})e^{-10^{10}t}$$

$$q(t) = 5\times 10^{-12}(1 - e^{-10^{10}t})$$

$$E_c = \frac{q(t)}{C} = 5(1 - e^{-10^{10}t})$$

We will need that $E_C = \frac{1}{C}q(t)$ from page 2 to find t for $E_C = 3$ volts.

$$3 = 5(1 - e^{-10^{10}t})$$

$$\frac{3}{5} = 1 - e^{-10^{10}t}$$

$$-\frac{2}{5} = -e^{-10^{10}t}$$

$$\frac{2}{5} = e^{-10^{10}t}$$

$$\ln(2/5) = \ln(e^{-10^{10}t})$$

$$\ln(2/5) = -10^{10}t \quad \ln e = 1$$

$$t = \frac{\ln(2/5)}{-10^{10}}$$

$$t \approx 9.2\times 10^{-11} \text{ seconds}$$

$$\text{amp} = \frac{\text{coulomb}}{\text{second}}$$

$$\frac{dq}{dt} = I \text{ (amps)}$$