We know how a function changes but do *not* know the function itself.

Differential Equations
Class Notes
Introduction and Background (Section 1.1)

Definition: A **differential equation** (diff. eq.) is an equation with derivatives of some *unknown* function. We will be finding this unknown function.

An example from calculus:
Think back to applications about the height, velocity, and acceleration of objects in free fall.

Let $h(t)$ represent the height of the object at time t. Then $h'(t)$ represents the object's velocity and $h''(t)$ is the object's acceleration.

Newton's Second Law tells us that $F = ma$ for the forces acting on an object. Since a is acceleration here, we can rewrite this as $m\left({d^2h}/{dt^2}\right) = -mg$ (m being the mass of the object). This simplifies to ${d^2h}/{dt^2} = -g$. How would we use calculus to then find $h(t)$?

This is a differential equation for an unknown function.

Let's integrate (twice) to find the function for the object's height.

We would get $\frac{dh}{dt} = -gt + c_1$ and $h = \frac{-gt^2}{2} + c_1 t + c_2$ (where c_1 and c_2 are real numbers).

Let's imagine if we had some information about this object, like its initial height and velocity, we could determine the constants c_1 and c_2.

Solving differential equations is essentially that process. The caveat is that *not* all equations are so easy to solve. We will learn many processes and tricks to decipher the unknown function from the information given about its derivatives.

We will explore many different applications besides Newtonian physics. Radioactive decay and fluid flow are among them. Problems that involve "rates of change" will often result in differential equations.

Main ideas:

1. The solution to a differential equation will be a function.
2. Integration is used lots!
3. The solution to a differential equation will *not* be unique because of constants of integration. We may be given initial conditions which nail down a unique solution.

Definitions: Dependent and **Independent Variables**: Recall, a function y is said to "be a function of x". This implies that the inputs are x-values and the outputs are y-values. In that context, y depends on x, so we say that x is the independent variable and y is the dependent variable.

A differential equation, like $\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right) + ky = 0$, implies that x is the independent variable and y is the dependent variable. Also, a and k are **coefficients**.

A differential equation, like $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y$, implies that x and y are both independent variables and u is the dependent variable.

Definition: Ordinary differential equation: a diff. eq. that has only ordinary derivatives with respect to a single independent variable.

Definition: Partial differential equation: a diff. eq. that contains partial derivatives with respect to more than one independent variable.

Definition: The **order** of a diff. eq. is the highest order derivative appearing in the equation.

Handouts: These are available through www.stlmath.com and will be given out in class. You can find others online.
Derivatives and Integrals Cheat Sheet (author, Paul Dawkins)
http://tutorial.math.lamar.edu/pdf/Common_Derivatives_Integrals.pdf

Many Integrals on a Single Page (site, integral-table.com) (Choose Fit to Page when printing)
http://integral-table.com/downloads/single-page-integral-table.pdf

Table of Basic Integrals (site, integral-table.com)
http://integral-table.com/downloads/Basic-Integral-Table.pdf

Definition: Linear differential equation: a diff. eq. in which the dependent variable y and its derivatives appear in additive combinations of their first powers. More precisely,

$$a_n(x)\left(\frac{d^n y}{dx^n}\right) + a_{n-1}(x)\left(\frac{d^{n-1} y}{dx^{n-1}}\right) + \cdots + a_1(x)\left(\frac{dy}{dx}\right) + a_0(x)y = F(x)$$

Notice, the functions $a_i(x)$ and $F(x)$ only depend on the variable x.

Differential equations that have higher powers of y or its derivatives are called **nonlinear**. These comprise most differential equations in practice. But we start with the simpler linear diff. eq.

Recall, the line $ax + by = c$ and the plane $ax + by + cz = d$ are such that they can be written as additive combinations of their first powers only.

expl 1: Can you tell why these diff. eq. are *not* considered linear? Also, please give the order of each diff. eq.

a.) $\frac{d^3y}{dx^3} + y^3 = 0$

Why is $t^3\frac{dx}{dt} = t^3 + x$ linear while the y^3 in part *a* was a problem?

b.) $\frac{d^2y}{dx^2} - y\left(\frac{dy}{dx}\right) = \cos x$

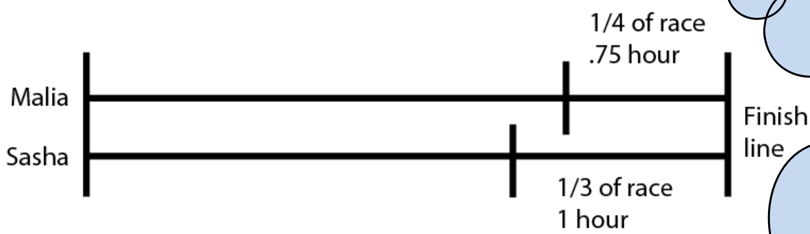
expl 2: Write a diff. eq. that fits the physical description.

The velocity at time t *of a particle moving along a straight line is proportional to the fourth power of its position* x.

Use k as the proportionality constant.

Do you remember $y = kx$ (directly proportional) and $y = k/x$ (inversely proportional)?

expl 3: The Obama daughters are competing in the biggest race of their lives. They started from a standing start and they each run with constant acceleration. After giving it her all, Malia covers the last ¼ of the distance in ¾ of an hour. Sasha covers the last $^1/_3$ of the distance in 1 hour. Who won and by how much time?



Let $s(t)$ be a runner's position at time t. We are told that a runner's acceleration is constant. Make a diff. eq. that shows this. Can you solve it for s (with no constants left over)?

Let's use L for the total length of the race.

We will use the fact that Malia's total time for the race equals the time for her to run the last ¼ of the race plus the time it takes to run the first ¾ of the race. We'll need to solve our position equation for t. For Malia, we'll call it t_M. Solve for $t_M(L)$ and do it for Sasha too.

extra room for calculations: