

We will solve a linear first-order diff. eq. with a special method that forces it into a form with which we can work.

Recall: Definition: A **linear first-order differential equation** is of the form

$a_1(x) \frac{dy}{dx} + a_0(x) \cdot y = b(x)$. Here, $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on x , not y . The **standard form** of a linear diff. eq. is $\frac{dy}{dx} + P(x) \cdot y = Q(x)$.

So, how do we solve them? We have two cases.

Methods for Solving Linear First-order diff. eq.:

Case 1: If $a_0(x) = 0$, then $a_1(x) \frac{dy}{dx} = b(x)$ and you can solve by solving for dy/dx and

integrating. This would get us $\frac{dy}{dx} = \frac{b(x)}{a_1(x)}$ and $y = \int \frac{b(x)}{a_1(x)} + c$ for some $c \in \mathbb{R}$. This assumes that $a_1(x)$ is *not* equal to zero.

This case is rare.

Case 2: If $a_0(x) = a_1'(x)$, then the diff. eq. $a_1(x) \frac{dy}{dx} + a_0(x) \cdot y = b(x)$ becomes

$a_1(x) \frac{dy}{dx} + a_1'(x) \cdot y = b(x)$. But do you recognize the left side?

This could be written $\frac{d}{dx}(a_1(x) \cdot y) = b(x)$. We can integrate this to solve for y , getting

$$y = \frac{\int b(x) dx + c}{a_1(x)}.$$

Case 2 seems like it would be rare too. But, it turns out that *any* linear 1st-order diff. eq. can be turned into a case 2 equation by multiplying by an “integrating factor”. We will call this factor $\mu(x)$.

The symbol μ is pronounced “mew”.

What we will essentially be doing is multiplying our whole equation by $\mu(x)$. If we choose this $\mu(x)$ correctly, that will turn our equation into the form we saw back in case 2. The book justifies why we use the $\mu(x)$ as defined below.

Method for Solving (Case 2) Linear First-order diff. eq.:

a.) Write the equation in the standard form $\frac{dy}{dx} + P(x) \cdot y = Q(x)$.

b.) Calculate $\mu(x) = e^{\int P(x)dx}$.

The constant of integration can be anything, so choose zero.

c.) Multiply the equation by $\mu(x)$.

This yields $\mu(x) \cdot \frac{dy}{dx} + \mu(x) \cdot P(x) \cdot y = \mu(x) \cdot Q(x)$. More importantly, we see this is equal to

$$\frac{d}{dx}(\mu(x) \cdot y) = \mu(x) \cdot Q(x).$$

The book shows how $\mu'(x) = \mu(x) \cdot P(x)$.

Focus on this last form.

In practice, use this last form here to solve the diff. eq. for y .

d.) Integrate both sides and divide by $\mu(x)$ to solve for y . This gets us $y = \frac{\int \mu(x)Q(x)dx + c}{\mu(x)}$.

When we calculate $\mu(x) = e^{\int P(x)dx}$, we can take the *easiest to use* function that satisfies $\mu'(x) = \mu(x) \cdot P(x)$.

This c comes from the integration.

Definition: General solution: This is the solution with the constant of integration above in place. We recently called this a **one-parameter family of solutions**.

expl 1: Obtain a general solution to the equation below.

$$\frac{dy}{dx} = \frac{y}{x} + 2x + 1$$

Follow steps *a*
through *d*. Label
them as you go.

Identify $P(x)$
and $Q(x)$.
Calculate $\mu(x)$.

You will get $\mu(x) = \frac{1}{|x|}$.

Can we use $\mu(x) = \frac{1}{x}$?

Do *not* forget
your constant
of integration
and solve for y .

Initial Value Problems:

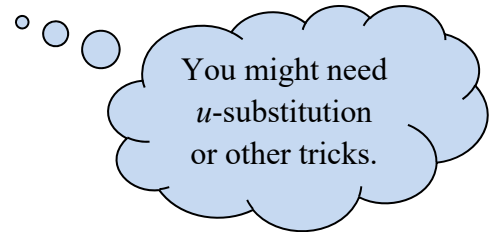
We combine this solution method with an initial value problem set-up and get the following theorem.

Theorem 1: Existence and Uniqueness of Solution:

If $P(x)$ and $Q(x)$ are continuous on an interval (a, b) that contains the point x_0 , then for any choice of initial value y_0 , there exists a unique solution $y(x)$ on (a, b) to the initial value

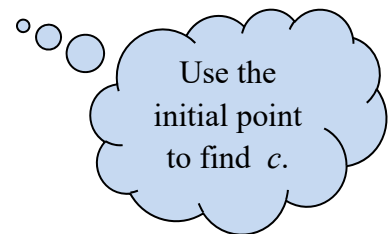
problem $\frac{dy}{dx} + P(x) \cdot y = Q(x), \quad y(x_0) = y_0.$

In fact, the solution is given by $y = \frac{\int \mu(x)Q(x)dx + c}{\mu(x)}$ for a suitable value of c .



expl 2: Solve the initial value problem.

$$\frac{dy}{dx} + 4y - e^{-x} = 0, \quad y(0) = \frac{4}{3}$$



expl 3: **Application: Secretion of Hormones:** The secretion of hormones into the blood is often a periodic activity. If a hormone is secreted on a 24-hour cycle, then the rate of change in the level of the hormone in the blood may be represented by the initial value problem

$\frac{dx}{dt} = \alpha - \beta \cos\left(\pi \cdot t / 12\right) - kx$, $x(0) = x_0$. Here, $x(t)$ is the level of the hormone in the blood at

time t , α is the average secretion rate, β is the amount of daily variation in the secretion, and k is a positive constant reflecting the rate at which the body removes the hormone from the blood.

If $\alpha = \beta = 1$, $k = 2$, and $x_0 = 10$, find $x(t)$.

Put it all in and try to write the diff. eq. in standard form.

Wacky! Integral formula #107

$$\int e^{bx} \cos(ax) dx = \frac{1}{a^2 + b^2} \cdot e^{bx} (a \sin(ax) + b \cos(ax))$$

(extra room for work)

Is this equation linear?

Sometimes an equation will *not appear* linear because we are thinking of the traditional roles of independent and dependent variables. We will see differential equations where, if we take the x to be the independent and y to be the dependent variables, it will *not* be linear. However, if we switch that and let the y be the independent variable, it can be shown to be linear. This does *not* happen in this section but will in the next.

An example is $\theta dr + (3r - \theta - 1)d\theta = 0$. We will explore this later. We will see that, if we take θ as the dependent variable, it is *not* linear. However, if we take r as the dependent variable, it can be shown to be linear.

Worksheet: Separable and Linear Differential Equations Practice:

This worksheet will give you a couple of diff. eq. to practice.