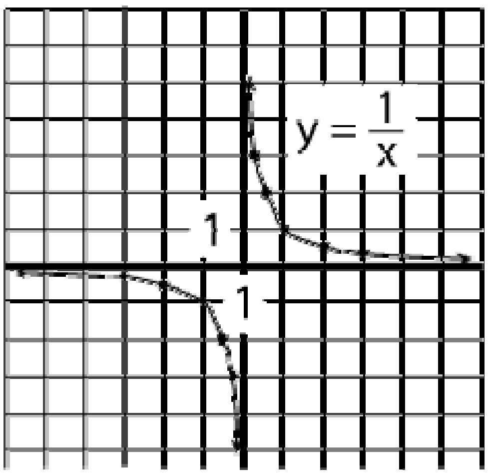
Limits as x approaches infinity NAME:

This worksheet will work on the notion of the limit of a function as x approaches negative or positive infinity. You can think of this limit as the one value the y values approach as we get further and further toward the left or right end of the graph. Remembering horizontal asymptotes of rational functions may be helpful; they will be referred to later.

1. Consider the graph of $y = \frac{1}{x}$ below. Pay particular attention to the far left and far right ends of the graph. What number are the y values getting closer and closer to? The answer is the same on both ends of the graph.



As you go further and further toward the left end of the graph, we say x approaches negative infinity. The number that the y values are approaching at this end of the graph will be called $\lim_{x \to -\infty} \frac{1}{x}$.

Likewise, as you go further and further toward the right end of the graph, we say x approaches positive infinity. The number that the y values are approaching at this end of the graph will be called $\lim_{x \to \infty} \frac{1}{x}$.

What are these limits equal to? In other words, what value do the y values approach?

2. This can also be seen by simply thinking about what happens to $\frac{1}{x}$ as x gets bigger and bigger. Complete the table for more evidence of this limit.

x	**value of $\frac{1}{x}$ (decimal form)**
10	
100	
1,000	
10,000	
1,000,000	

3. Horizontal asymptotes of rational functions are essentially this idea. In the case of rational functions $f(x)$, the $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ will be the same number. To figure horizontal asymptotes of rational functions, and therefore these limits, recall the following.

A rational function is a function that can be written as a fraction where the top and bottom are both polynomial functions. (Recall that the degree of a polynomial function is the highest exponent of the x's. The leading term of a polynomial is the term that contains this highest exponent.)

There are three cases:

1. degree on top is **equal to** degree on bottom,
2. degree on top is **less than** degree on bottom, and
3. degree on top is **greater than** degree on bottom.

For case 1, the horizontal asymptote is gotten by dividing the leading terms on top and bottom. For case 2, the horizontal asymptote is always $y = 0$. The function $y = \frac{1}{x}$ is an example of this. For case 3, the (oblique) asymptote is the quotient gotten by dividing the bottom into the top. In this last case, we would say the limit does not exist, since the y values will not approach a single value.

When we are finding limits, we will run into rational functions and have to determine which of these three cases apply. For the most part, we will be dealing with the second case.

4. We will also need to find limits of constant functions like $y = 5$. These constant functions are just horizontal lines. Draw a quick graph of $y = 5$ and ask yourself what value do the *y* values approach as *x* gets closer and closer to negative or positive infinity. This number is the limit of 5 as *x* approaches negative or positive infinity and is denoted by $\lim_{x \to -\infty} 5$ or $\lim_{x \to \infty} 5$.

5. Taking this specific example into consideration, what would $\lim_{x \to -\infty} c$ or $\lim_{x \to \infty} c$ where *c* is any real number?

If a specific number exists such that the *y* values approach it on the left or right end of the graph, that limit exists. If the *y* values soar off toward infinity or negative infinity, the limit does not exist.

For some functions, $\lim_{x \to -\infty} f(x)$ exists but $\lim_{x \to \infty} f(x)$ does not, or vice versa. Examples of such functions are $y = x$ or $y = e^x$. Graph these if you cannot picture them in your head to help you answer the next two questions.

6. Investigate the graph of $y = x$ to find $\lim_{x \to -\infty} x$ and $\lim_{x \to \infty} x$.

7. Investigate the graph of $y = e^x$ to find $\lim_{x \to -\infty} e^x$ and $\lim_{x \to \infty} e^x$.

Find the limits listed below. You can use your knowledge of horizontal asymptotes in the cases where rational functions are concerned. You could also graph the functions and see what value the *y* values approach as *x* approaches negative or positive infinity, whichever is indicated by the limit.

8. $\lim_{b\to\infty} \frac{-1}{b}$

9. $\lim_{a\to-\infty} \frac{1}{2(a-2)^2}$

10. $\lim_{b\to\infty} b$

11. $\lim_{x\to\infty} \frac{x^2}{3x^2}$

12. $\lim_{x\to-\infty} e^{2x}$

More complicated limits must be looked at piecemeal. We must remember the limit rules we learned when we studied limits before such as $\lim_{x\to\infty}\left(c * f(x)\right) = c * \lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty}\left(f(x) \pm g(x)\right) = \lim_{x\to\infty} f(x) \pm \lim_{x\to\infty} g(x)$.

Consider finding $\lim_{x\to\infty}\left(\frac{-1}{2(\ln x)^2} + \frac{1}{2}\right)$. We think first about what happens to $y = \ln x$ as x approaches infinity. The y values slowly but surely approach infinity as well. Since this is on the bottom of a fraction, the fraction $\frac{-1}{2(\ln x)^2}$ will approach 0. (This is because a number divided by increasingly bigger and bigger numbers will approach 0 as $\frac{1}{x}$ did.) The $1/2$ is a constant so its limit is itself. Hence $\lim_{x\to\infty}\left(\frac{-1}{2(\ln x)^2} + \frac{1}{2}\right) = \frac{1}{2}$.

Find the limits below. Remember you can always just graph the function in question. However, it is helpful to be able to think through them. To think through them, it helps to have pictures of the basic functions like $y = e^x$ and $y = \ln x$ in your head.

13. $\lim_{x\to\infty}\left(\ln(x+2)+3\right)$

14. $\lim_{x\to\infty}\left(\frac{3}{e^x}\right)$

15. $\lim_{x\to-\infty}\left(\frac{3}{e^x}\right)$

16. $\lim_{x\to-\infty}\left(\frac{e^x}{3x+4}\right)$

17. $\lim_{a\to-\infty}\left(\frac{-1}{4}+\frac{1}{4\left(1+a^4\right)}\right)+\lim_{b\to\infty}\left(\frac{-1}{4\left(1+b^4\right)}+\frac{1}{4}\right)$