

We have rules to help us find the derivatives of the product or quotient of two functions. Economics lends an example.

Business Calculus
Class Notes

Section 1.6: The Product and Quotient Rules with Average Cost, Revenue, and Profit

Keep in mind that the derivative of a function is its rate of change.

Consider a person running down the street. Their running speed would be equal to the product of their stride length and their stride rate. For instance, if your stride length is three feet per stride and your stride rate is two strides per second, what is your running speed? Include units.

$$\text{running speed} = \frac{3 \text{ ft}}{\cancel{\text{stride}}} \cdot \frac{2 \cancel{\text{strides}}}{1 \text{ sec}} = 6 \text{ ft/sec}$$

Now, suppose your stride length increases to 3.5 feet per stride while your stride rate remains at two strides per second, what is your running speed? Include units.

$$\text{running speed} = \frac{3.5 \text{ ft}}{\cancel{\text{stride}}} \cdot \frac{2 \cancel{\text{strides}}}{1 \text{ sec}} = 7 \text{ ft/sec}$$

We could calculate the **change in speed** by subtracting the two answers above. However, we could also multiply the change in stride length by the stride rate. Do this now, including units.

$$\text{change in running speed} = \frac{0.5 \text{ ft}}{\cancel{\text{stride}}} \cdot \frac{2 \cancel{\text{strides}}}{1 \text{ sec}} = 1 \text{ ft/sec}$$

Alternatively, imagine your stride length is still three feet per stride but your stride rate increases to 2.25 strides per second. Find your change in speed by multiplying the stride length by the change in stride rate. Include units.

$$\text{change in running speed} = \frac{3 \text{ ft}}{\cancel{1 \text{ stride}}} \cdot \frac{0.25 \cancel{\text{strides}}}{1 \text{ sec}} = 0.75 \text{ ft/sec}$$

We have seen the result of changing one of the factors. But what happens if we change both stride length and stride rate? You can expect a change in speed of $1 + 0.75 = 1.75$ feet per second.

This leads us to the **Product Rule**. As we look at it on the next page, imagine $f(x)$ to be stride length and $g(x)$ to be stride rate. To find the **rate of change** (in other words, the **derivative**) in running speed (which is the product of $f(x)$ and $g(x)$), we have to do some maneuvering.

THEOREM 3.6 Product RuleIf f and g are differentiable at x , then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Again, this theorem comes from another book.

We will use it to find the derivatives of products of functions.

expl 1: Find these derivatives two different ways. First, find the derivative using the new product rule. Second, find the derivative by multiplying the function out and then using the rules from the previous section. Simplify each until you can see they match.

a.) $h(x) = 3x^4(2x^2 - 1)$

Product Rule

$$h'(x) = 12x^3(2x^2 - 1) + 3x^4(4x)$$

$$= 24x^5 - 12x^3 + 12x^5$$

$$h'(x) = 36x^5 - 12x^3$$

OR Multiply it out, then Derivate

$$h(x) = 6x^6 - 3x^4$$

$$h'(x) = 6 \cdot 6x^5 - 3 \cdot 4x^3$$

$$h'(x) = 36x^5 - 12x^3$$

b.) $f(t) = (3\sqrt{t} - 5t)(4t^3)$

Product Rule:

$$f(t) = \overset{\text{1st factor}}{(3t^{1/2} - 5t)} \overset{\text{2nd factor}}{(4t^3)}$$

$$f'(t) = \left(\frac{3}{2}t^{-1/2} - 5\right)(4t^3) + (3t^{1/2} - 5t)(12t^2)$$

$$= \frac{12}{2}t^{5/2} - 20t^3 + 36t^{5/2} - 60t^3$$

$$f'(t) = 42t^{5/2} - 80t^3$$

Multiply, then Derivate: $f(t) = 12\sqrt{t} \cdot t^3 - 20t^4 = 12t^{1/2}t^3 - 20t^4$

$$f(t) = 12t^{7/2} - 20t^4$$

$$f'(t) = 12 \cdot \frac{7}{2}t^{5/2} - 20 \cdot 4t^3$$

$$f'(t) = 42t^{5/2} - 80t^3$$

We have the Quotient Rule because of the Product Rule. I provide it here without proof.

THEOREM 3.7 Quotient Rule

If f and g are differentiable at x and $g(x) \neq 0$, then the derivative of f/g at x exists and

$$\star \frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

expl 2: Find the derivative of this quotient.

$$h(x) = \frac{x^3 - 4x^2 + x}{x - 2}$$

$\leftarrow f(x)$
 $\leftarrow g(x)$

$$h'(x) = \frac{(3x^2 - 8x + 1)(x - 2) - (x^3 - 4x^2 + x)(1)}{(x - 2)^2}$$

$$h'(x) = \frac{3x^3 - 8x^2 + x - 6x^2 + 16x - 2 - x^3 + 4x^2 - x}{(x - 2)^2}$$

$$h'(x) = \frac{2x^3 - 10x^2 + 16x - 2}{(x - 2)^2}$$

$$h'(x) = \frac{2x^2(x - 5) + 2(8x - 1)}{(x - 2)^2}$$

OR

$$\frac{LoD(Hi) - HiD(Lo)}{(Lo)^2}$$

$$w^{-1/2} w^{1/2} = w^0 = 1$$

expl 3: Find the derivative of this quotient and simplify. (I like to write all instances of $\sqrt{w}$ as $w^{1/2}$.)

$$g(w) = \frac{\sqrt{w} + w}{\sqrt{w} - w} \quad g(w) = \frac{w^{1/2} + w}{w^{1/2} - w}$$

FoIL and distribute minus sign

$$g'(w) = \frac{(\frac{1}{2}w^{-1/2} + 1)(w^{1/2} - w) - (w^{1/2} + w)(\frac{1}{2}w^{-1/2} - 1)}{(w^{1/2} - w)^2}$$

$$= \frac{\frac{1}{2} + \underline{w^{1/2}} - \frac{1}{2}w^{1/2} - w - \frac{1}{2} - \frac{1}{2}w^{1/2} + \underline{w^{1/2}} + w}{(w^{1/2} - w)^2}$$

$$g'(w) = \frac{1 \cdot w^{1/2}}{(w^{1/2} - w)^2} \quad \checkmark$$

expl 4: You will encounter problems that use both rules. Find this derivative. (End with a single fraction. You do *not* need to multiply out the bottom.)

$$h(x) = (x^3 + 5) \cdot \frac{x^2 + 16}{x^2 - 16}$$

$$\rightarrow \text{Find } \frac{d}{dx} \left(\frac{x^2 + 16}{x^2 - 16} \right) = \frac{2x(x^2 - 16) - (x^2 + 16) \cdot 2x}{(x^2 - 16)^2}$$

$$= \frac{2x^3 - 32x - 2x^3 - 32x}{(x^2 - 16)^2}$$

Product Rule:

$$\frac{d}{dx} \left(\frac{x^2 + 16}{x^2 - 16} \right) = \frac{-64x}{(x^2 - 16)^2}$$

$$h'(x) = \frac{(3x^2)}{1} \cdot \frac{x^2 + 16}{x^2 - 16} + \frac{(x^3 + 5)}{1} \cdot \frac{-64x}{(x^2 - 16)^2}$$

$$= \frac{3x^2(x^2 + 16)(x^2 - 16)}{(x^2 - 16)^2} + \frac{-64x(x^3 + 5)}{(x^2 - 16)^2}$$

$$h'(x) = \frac{3x^2(x^2 + 16)(x^2 - 16) - 64x(x^3 + 5)}{(x^2 - 16)^2}$$

Rates of Change, in General:

We are working with specific examples here but understand that these ideas will work for any function. Let's say you know a function $p(t)$ measures something like a population or price per item or number of internet users, etc. Here, t is taken to be time where $t \geq 0$.

This function's **instantaneous rate of change** is its derivative $p'(t)$ or $\frac{dp}{dt}$. This measures how fast $p(t)$ is changing with respect to time. The units of this rate of change will always be "units of $p(t)$ divided by units of t ".

Average Cost, Revenue, and Profit and Their Rates of Change:

Often, we know a company's cost and revenue based on the number of items they produce. We can find the company's profit.

Recall that profit equals revenue minus cost.

Let $C(x)$ be the cost of producing x items. Perhaps we will be given the function $C(x) = 3.4x^{0.6} + 1200$ dollars.

Pronounced "C bar".

★ The **Average Cost per Item** would be given by $\bar{C}(x) = \frac{C(x)}{x}$.

Further, let $R(x)$ denote the company's revenue if x items are sold. Perhaps we will be given the function $R(x) = 7.5x^{0.8}$ dollars.

This should make sense if you picture a total cost of \$500 for 100 items.

This leaves us **Profit** (the money we really make) as

$$P(x) = R(x) - C(x) \quad \star$$

It is *not* a long stretch to define average revenue and profit as you see here.

We are in the US so our units for average cost, revenue, and profit will usually be in dollars per unit.

★ The **Average Revenue per Item** would be given by $\bar{R}(x) = \frac{R(x)}{x}$.

★ The **Average Profit per Item** would be given by $\bar{P}(x) = \frac{P(x)}{x}$.

In a later section, we will explore **marginal** cost, revenue, and profit. These, we will see, are the derivatives of the cost, revenue, and profit functions. Today, we will be finding the derivatives of the average cost, revenue, and profit functions.

$$\frac{a^n}{a^m} = a^{n-m}$$

expl 5: A company has a cost function of $C(x) = 3.4x^{0.6} + 1200$ dollars when they produce x items. The company's revenue is given by $R(x) = 7.5x^{0.8}$ with x defined as before.

a.) Find the profit function $P(x)$.

$$P(x) = R(x) - C(x) \\ = 7.5x^{0.8} - (3.4x^{0.6} + 1200)$$

b.) Find the average profit function.

$$P(x) = 7.5x^{0.8} - 3.4x^{0.6} - 1200$$

$$\bar{P}(x) = \frac{P(x)}{x} = \frac{7.5x^{0.8} - 3.4x^{0.6} - 1200}{x}$$

$$\bar{P}(x) = 7.5x^{-0.2} - 3.4x^{-0.4} - 1200x^{-1}$$

c.) Find the rate at which the average profit is changing when $x = 100$ items are sold. Interpret.

$$\bar{P}'(x) = 7.5(-0.2)x^{-1.2} - 3.4(-0.4)x^{-1.4} - 1200(-1)x^{-2}$$

$$\bar{P}'(x) = -1.5x^{-1.2} + 1.36x^{-1.4} + 1200x^{-2}$$

$$\bar{P}'(100) = -1.5(100)^{-1.2} + 1.36(100)^{-1.4} + 1200(100)^{-2}$$

$$\bar{P}'(100) \approx \$0.12/\text{item}$$

The rate of change for a function is its derivative.

When we have sold 100 items, the average profit is changing by this amount for each additional item sold.

A positive rate of change means the function is increasing. It might be called a **growth rate**.

A negative rate of change means the function is decreasing. It might be called a **decay or decline rate**.

Slope = rise/run

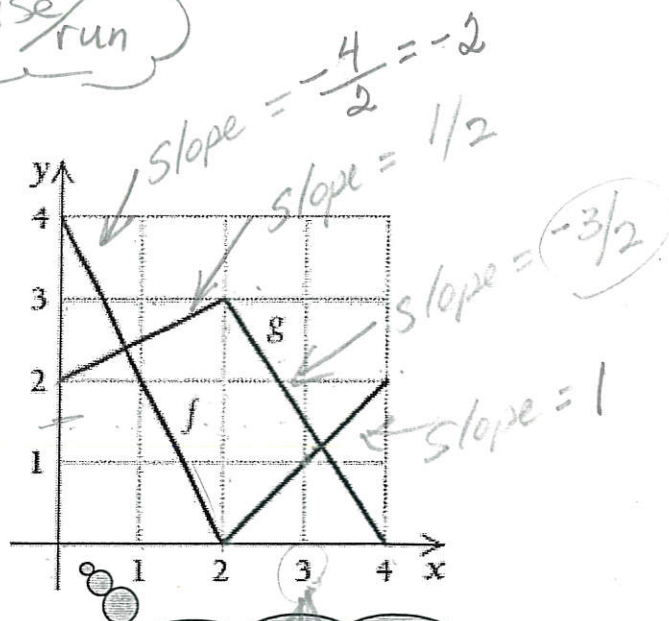
expl 6: Use the graphs shown here for the functions f and g to find the following values.

a.) $(f \cdot g)'(3)$

b.) $\left(\frac{f}{g}\right)'(2)$

c.) $(f \cdot g)'(1)$

Remember that $f \cdot g$ means $f(x) \cdot g(x)$ and we have a formula for its derivative.



Begin by finding the slopes of the four segments that comprise the graphs.

a) $(f \cdot g)'(3)$

$$= \underline{f'(3)} \cdot g(3) + \underline{f(3)} \cdot g'(3)$$

$$= 1 \cdot 1.5 + 1 \cdot (-3/2)$$

$$= 1.5 - 1.5$$

$(fg)'(3) = 0$

b) (do on your own)

$$(f/g)'(2) = \frac{\overset{\text{dne}}{f'(2)} g(2) - \overset{\text{dne}}{f(2)} g'(2)}{(g(2))^2}$$

The func (f/g) is not differentiable at $x=2$.

$(f/g)'(2)$ does not exist.

