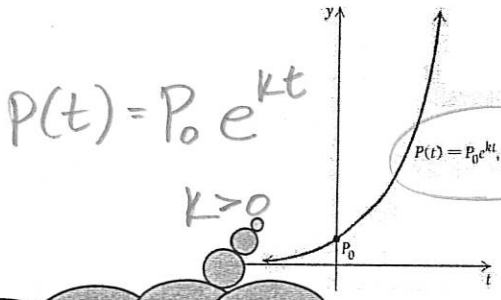
Knowing the rate of growth of a function, can we figure out its formula?

Business Calculus
Class Notes
Section 2.4: Applications: Uninhibited and Limited Growth Models

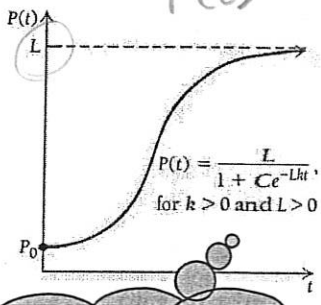
We are going to study two functions which can be used to model things from population growth to the cost of a postage stamp to money in a bank account. The two types of functions have the following graphs.

$P(t) = P_0 e^{kt}$, $k > 0$



This first one is basic (uninhibited) **exponential growth.**

$P(t) = \frac{L}{1 + Ce^{-Lkt}}$, $k > 0$, $L > 0$



This is called a **logistic function** and models **limited growth.**

We will often start with an equation that tells us an unknown function's rate of change or derivative. These equations are called **differential equations**. The solution to such an equation is the unknown function.

To get us started, find the derivative of $y = 5e^{3x}$. Let's call it $\frac{dy}{dx}$. Don't forget the Chain Rule.

$g(x) = 3x \rightarrow g'(x) = 3$

$\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \cdot g'(x)$ (from 2.2)

$\frac{dy}{dx} = 5e^{3x} \cdot 3$

This is y.

So, $\frac{dy}{dx} = y \cdot 3$

Do you see how $\frac{dy}{dx} = 3y$?

It so happens that the *only* function whose derivative is equal to some constant times itself is that special exponential function we have been studying. The theorem that states this is on the next page.

Theorem 9

A function $y = f(x)$ satisfies

$$\frac{dy}{dx} = ky \quad \text{or} \quad f'(x) = k \cdot f(x)$$

if and only if

$$y = ce^{kx} \quad \text{or} \quad f(x) = ce^{kx}. \quad \text{where } c \in \mathbb{R}$$

This k is the **rate of exponential growth**. Notice this is *not* the rate of change for the function, which is k times the function.

expl 1: Find f if $f'(x) = 4f(x)$. Check your answer.

$k = 4$

Thm 9 tells us

$f(x) = ce^{4x}$

Check:

$f(x) = ce^{4x}$

$f'(x) = ce^{4x} \cdot 4$

$= 4 \cdot f(x)$ ✓

expl 2: A pizza company is selling franchise rights. It is estimated that the number of franchises, N, will increase at the rate of 10% per year. In other words, $\frac{dN}{dt} = 0.10N$ where t is the number of years after some starting point. Answer the following questions.

$k = 0.10$

a.) Find the function that satisfies this equation. Assume the number of franchises at $t = 0$ is 50 franchises.

$N = ce^{0.10t}$

$t = 0$, $N = 50$

$50 = ce^{0.10(0)}$

$50 = ce^{0}$

$50 = c$

$N = 50e^{0.10t}$

This bit of information will help us find c.

b.) How many franchises will there be in 20 years?

$N = 50e^{0.10(20)}$

≈ 369 franchises

c.) What is the rate of change in the number of franchises in 20 years? Include units.

$N = 50e^{0.10t}$

$N' = 50e^{0.10t} \cdot 0.10$

$N' = 5e^{0.10t}$

$N'|_{t=20} \approx 37$ franchises/year

The Exponential Model for Uninhibited Growth:

This graphic from the book restates Theorem 9 and goes on to define all the bits. You'll also see an alternative form at the very bottom. This assumes P is a population and we see P_0 used as the initial population (population at time $t = 0$).

The differential equation

$$\frac{dP}{dt} = kP \quad \text{or} \quad P'(t) = kP(t), \quad \text{with } k > 0,$$

is the model of uninhibited (unrestrained) population growth. In the absence of inhibiting or stimulating factors, a population normally reproduces at a rate proportional to its size, and this is exactly what $dP/dt = kP$ says. The only function that satisfies this differential equation is given by

$$P(t) = ce^{kt},$$

where t is time and k is the rate expressed in decimal notation. Note that

$$P(0) = ce^{k \cdot 0} = ce^0 = c \cdot 1 = c,$$

so c represents the initial population, which we denote as P_0:

$$P(t) = P_0 e^{kt}.$$

$P(t)$ = population at time t

P_0 = initial population

k = rate of growth.

$I(0) = 52{,}840$ $I(3) = 56{,}430$

expl 3: In 2012, the US per capita personal income was \$52,840. In 2015, it had grown to \$56,430. Assume that the growth of personal income per capita follows an exponential model.

Sometimes we are *not* given k. We must assume $I(t) = ce^{kt}$ and find k.

a.) Find the function that estimates this income $I(t)$, in dollars, t years after 2012. (We'll find c and then k.)

$I(t) = ce^{kt}$

$I(t) = 52840\,e^{kt}$

$56430 = 52840\,e^{k(3)}$

$\frac{56430}{52840} = \frac{52840e^{3k}}{52840}$

$1.07 \approx e^{3k}$

$\ln 1.07 = \ln e^{3k}$

$\ln 1.07 = 3k \cdot \ln e$

$k \approx \frac{\ln 1.07}{3}$

$k \approx 0.0219$

$I(t) = 52840\,e^{0.0219t}$

see 2.1 notes

b.) Find the rate of change (or derivative) of this function in 2025.

We know $I'(t) = k \cdot I(t)$.

$I(t) = 52840\,e^{0.0219t}$

$I'(t) = 52840 \cdot e^{0.0219t} \cdot (0.0219)$

$I'(t) = 1157.196\,e^{0.0219t}$

In 2025, $t = 13$

$I'(13) \approx \$1538/\text{year}$

That's all well and good but many populations and other variables do *not* follow an uninhibited growth model. Think of a deer population which may grow and grow and grow. But, at some point, they start running out of resources like food, and their population does *not* keep growing at the same rate. We call these models **limited**. We will explore one such model and only touch on another.

The Logistic Model for Limited Growth:

The differential equation

$$\frac{dP}{dt} = kP(L - P)$$

is the model for the **logistic function**, where L is the upper limit of the population P. Here, the rate of change of the population is directly proportional to the population P and to the remaining room for population growth $L - P$. The function that solves this differential equation has the simplified form

$$P(t) = \frac{L}{1 + Ce^{-Lkt}}, \quad \text{or} \quad P(t) = \frac{L}{1 + Ce^{-rt}},$$

exponent $= -rt$

exponent $= -Lkt$

where $r = Lk$ and both $k > 0$ and $L > 0$.

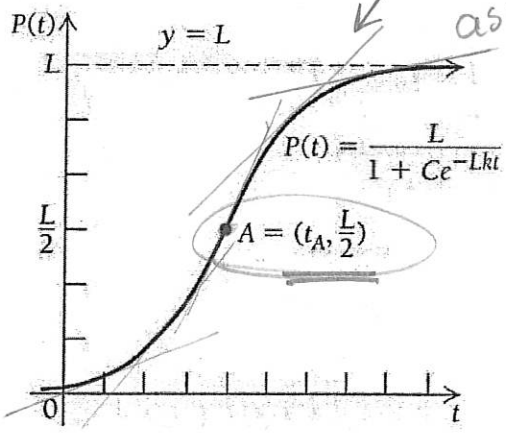
Here's another picture from the book. Notice the book has marked a point A and the line $y = L$.

The limiting value L is also called the **carrying capacity**. It's a horizontal asymptote.

Point A is called the **point of diminishing returns**. It is here where the rate of change of P peaks and starts to slow down. Take note that the y-value of this point is simply $L/2$.

This function starts off like an exponential function but then the limited resources slow the growth rate.

horizontal asymptote



Imagine some tangent lines along this graph.

When given such a function, you can find its derivative by the differential equation at the top of this page or just use your basic rules of differentiation.

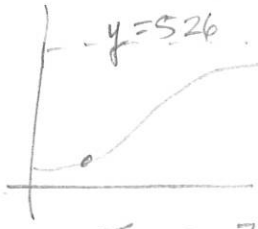
(from pg 5) $P(t) = \dfrac{L}{1+Ce^{-Lkt}}$

expl 4: Judd's Suds has advertised online. The number of views of an ad, in thousands, after t hours is given by $V(t) = \dfrac{526}{1+4.7e^{-0.03t}}$. Answer the following questions.

a.) How many views were there after 20 hours? After 200 hours? Label appropriately.

$V(20) \approx 147{,}000$ views

$V(200) \approx 520{,}000$ views



$[0, 200] \times [0, 550]$

Graph the function. Find L in the formula to help set *ymax*. Use the Value function for part *a*.

b.) Find the rate of change in the number of views. In other words, find $V'(t)$.

from pg 5: $\dfrac{dP}{dt} = kP(L-P)$

Our V(t) is this P.

$V'(t) = \dfrac{5.70 \times 10^{-5}}{1}\left(\dfrac{526}{1+4.7e^{-0.03t}}\right)\left(526 - \dfrac{526}{1+4.7e^{-0.03t}}\right)$

$L = 526$

To find k: exponents equal

$-0.03t = -Lkt$

$-0.03 = -Lk$

$-0.03 = -526k$

$5.70 \times 10^{-5} \approx k$

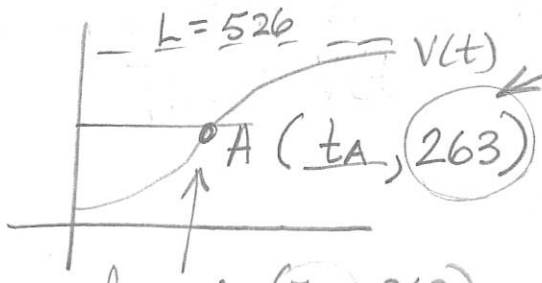
$V'(t) = \dfrac{0.03}{1+4.7e^{-0.03t}}\left(526 - \dfrac{526}{1+4.7e^{-0.03t}}\right)$

$V'(t) = \dfrac{15.78}{1+4.7e^{-0.03t}} - \dfrac{15.78}{(1+4.7e^{-0.03t})^2}$

c.) What is the number of views at the point of diminishing returns? Interpret.

Solve graphically.

A is (point of diminishing returns) and has y-value of $L/2 = 526/2 = 263$



found A: (52, 263) ← rounded

The point of diminishing returns occurs at 263,000 views. This occurs when $t \approx 52$ hours. This point is where $V'(t)$ stops increasing and starts decreasing.

The Limited Growth Function:

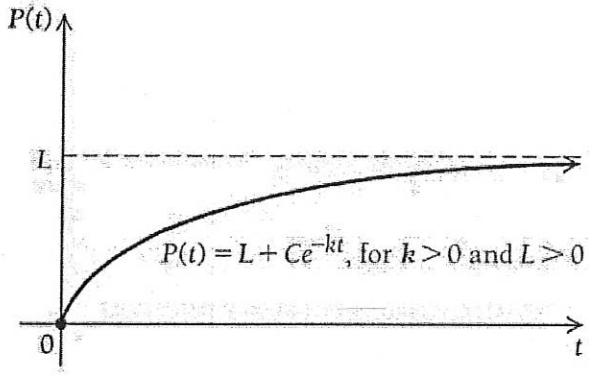
This one will *not* be represented in the homework. It is provided here for reference. Its use is similar to what we did in the last example.

The differential equation

$$\frac{dP}{dt} = k(L - P)$$

models the **limited growth function**, in which L is the upper limit of the population, P, and the rate of change of P is directly proportional only to the remaining room for population growth, $L - P$. The solution of this equation is

$$P(t) = L + Ce^{-kt}, \quad \text{where} \quad k > 0 \text{ and } L > 0.$$



Notice this one also models **limited growth**. Do you see why?

Check out the slopes of tangent lines (rates of change) and you'll see that they are greatest at the beginning and decrease as you go to the right. The domain is $[0, \infty)$.