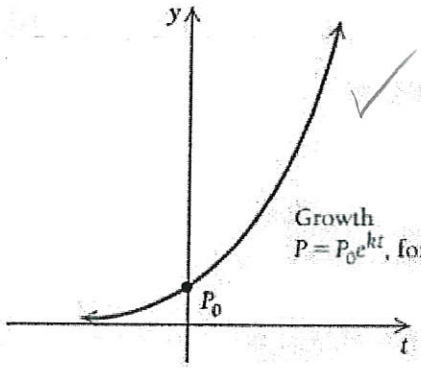
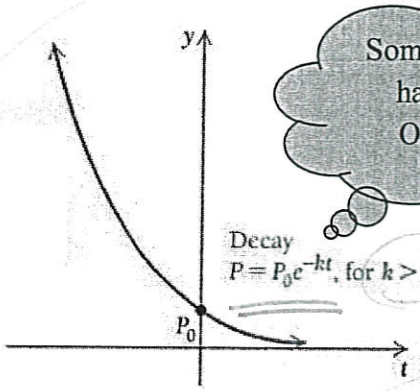
We explored exponential functions that increased in the last section. You know that they also may decrease.

Business Calculus
Class Notes
Section 2.5: Application: Exponential Decay

As we have seen, exponential functions can increase over time or they can decrease over time. Here are the pictures from the book with the formulas we'll use.





Some exponents will have a negative. Others will *not*.

$P = P_0e^{-kt}$

The left picture is of exponential growth and the right one shows exponential decay. Different sources may use different definitions but notice our book changes the formula itself to show when the function is decreasing (decay).

Using exponential decay models, we can explore topics such as the amount of a radioactive substance like carbon-14 or uranium, population decline, number of tuberculosis cases, and even the amount of money needed now so that it grows with interest to a larger amount in 20 years.

We'll need a good deal of algebra and our handy friend, the derivative.

expl 1: Iodine-131 has a decay rate of 9.6% per day. The rate of change of an amount N of iodine-131 after t days is given by $\frac{dN}{dt} = -0.096N$. Answer the following questions.

a.) Let N_0 represent the amount present at $t = 0$. Find the exponential function that models the decay.

$N(t) = N_0e^{kt}$ where $k = -0.096$

$N(t) = N_0e^{-0.096t}$

Think back to the previous section.

b.) Suppose 500 grams is present at time $t = 0$. How much will remain after 10 days? Include units.

$N(t) = 500e^{-0.096t}$

$N(10) = 500e^{-0.096(10)}$

≈ 191.4 grams

from 2.2: $\frac{d}{dx}(e^{f(x)}) = f'(x) \cdot e^{f(x)}$

expl 1 continued:

c.) What is the rate of change of the amount of iodine-131 after 10 days? Include units.

$N(t) = 500\,e^{-0.096t}$

$N'(t) = 500(-0.096)\,e^{-0.096t}$

$N'(t) = -48\,e^{-0.096t}$

$N'(10) = -48e^{-0.096(10)} \approx -18.4$ g/day

Half-life of Radioactive Substances:

How long will it take for a radioactive substance to decay so that it has exactly one-half of the original mass? This is called the **half-life of a substance** and varies depending on the element you have. Let's find this value in general as good algebraic practice.

We'll start with $P(T) = P_0 e^{-kT}$. We're using T to denote the half-life as opposed to t which is time in general.

We let P_0 denote the original amount. What should we substitute for $P(T)$?

$\frac{1}{2}$ of P_0

$\frac{1}{2}P_0 = P_0 e^{-kT}$

$\frac{1}{2} = e^{-kT}$

$\ln(1/2) = \ln e^{-kT}$

$\ln(1/2) = -kT \cdot \ln e$ (ln e → 1)

$\ln(2^{-1}) = -kT$

$-1 \cdot \ln 2 = -kT$

$\ln 2 = kT$ ✓

Just in case, here it is from the book.

Theorem 10

The decay rate, k, and the half-life, T, are related by

$$kT = \ln 2, \quad \text{or} \quad k = \frac{\ln 2}{T} \quad \text{and} \quad T = \frac{\ln 2}{k}.$$

Careful with the value of k.

expl 2: Concerning the iodine-131 from example 1, how long will it take for one-half of the original amount to decay? In other words, what is the half-life of iodine-131? Include units.

$N(t) = N_0 \cdot e^{-0.096t}$ or $500\,e^{-0.096t}$

Compare to $P = P_0 e^{-kt}$ from page 2.

So, $k = 0.096$ (positive!)

$\downarrow$

$T = \ln 2 / k$ (Thm 10)

$T = \dfrac{\ln 2}{0.096} \approx 7.22$ days

You will see varying applications that use this same setup. For examples 1 and 2, we looked at the amount of a radioactive substance. However, a problem about the decline of a population will look nearly identical and can be done with very similar work.

This next example will give us T? Can you see how? We will use it to quickly find k.

expl 3: The population of Cortez Breaks *decreases* at a rate proportional to the current size of the population. In 2017, the population was 5,000. By 2023, the population had decreased to 2,500.

a.) Write a formula for the population of Cortez Breaks t years after 2017.

$T = 6$ yrs

$k = \ln 2 / T$

$k = \ln 2 / 6$

$k \approx 0.1155$

$P(t) = 5000\,e^{-0.1155t}$

Get used to that meaning $\dfrac{dP}{dt} = -kP$.

In algebra, we are told that "y is (directly) proportional to x" means $y = k \cdot x$.

(from 2.4)

$P(t) = c\,e^{-kt}$

$\Downarrow$

$P(t) = P_0\,e^{-kt}$

$P(t) = 5000\,e^{-kt}$

b.) When will the population be 30% of the 2017 population?

$P(t) = 5000\,e^{-0.1155t}$

$1500 = 5000\,e^{-0.1155t}$

$0.30 = e^{-0.1155t}$

$\ln 0.30 = \ln e^{-0.1155t}$

$\ln 0.30 = -0.1155t \cdot \ln e$ ($\ln e = 1$)

$\ln(0.30) = -0.1155t$

$\rightarrow t = \dfrac{\ln(0.30)}{-0.1155}$

$t \approx 10.4$ yrs (after 2017)

$\downarrow$

Year 2027

Present Value:

Let's say you need \$50,000 but you have 10 years to get there. You'll invest money now so that it compounds continuously for those 10 years and grows to \$50,000. The amount you would need to invest now is called the **present value**. (This was mentioned in the Review of Exponential and Logarithmic Notes but was also called principal.)

Here is the formula the book gives us. If you compare it to what we saw in the Review of Exponential and Logarithmic Notes, you'll notice they changed around some variables so don't look too closely at that.

2.1

Theorem 11

The present value P_0 of an amount P due t years later, at interest rate k, compounded continuously, is given by

$$P_0 = Pe^{-kt}.$$

expl 4: An actor will be paid \$13 million in three years once a film has been completed. Find the present value of this amount, given it would be invested to be compounded continuously at 3.6%. In other words, what amount would be invested now so that it would be worth \$13 million in three years?

$P_0 = Pe^{-kt}$ $\quad K = 0.036$

$P_0 = 13e^{-0.036(3)}$

$P_0 \approx 11.669159$ millions of dollars

↓ OR

\$ 11,669,159

As we have seen, this percent in decimal form will be k.

Play around with using 13 or 13,000,000.

It's fun to think of all the various applications that this method helps us explore such as car depreciation, the amount of learning a student retains, and even the cooling temperatures of a cup of coffee. Start with the basics, be sure you know what the variables represent, and have fun.