Many exponential relationships do *not* use a base of e. We will now look at derivatives for a generic base a which is said to be positive but *not* 1.

Business Calculus
Class Notes

Section 2.6: Derivatives of Exponential and Logarithmic Functions with Base a

From the original definition of the exponential function $f(x) = a^x$, we will take that the base a has to be a positive number that is *not* equal to 1. Its inverse, $f^{-1}(x) = \log_a x$, inherits that characteristic. And off we go!

Let's say we are told that a population increases by two-fold (or doubles) in 3 years. Or, a bank account's value increases by five-fold (is quintuple a word?) over 10 years. Maybe the value of an appliance will be worth 1/3 of its value in 7 years. These are exponential functions and can be found by the following formula.

Base-n Exponential Growth

If a quantity increases n-fold over a fixed period of time T, then the quantity $P(t)$ at time t is given by

$$P(t) = P_0 \cdot n^{t/T},$$

where P_0 is the initial quantity.

exponent t/T

If n is less than 1, we have exponential decay and the quantity decreases.

We can find the equivalent function using base e. That will require the algebra of solving exponential equations. You will see that involves taking the natural log of both sides of the equation.

$F_0 = 75$ $\quad n = 3$ $\quad T = 8$ months

expl 1: Tonya had 75 followers on her blog. After an appearance on the Today show, the number of her followers started to triple every 8 months. Answer the following questions.

a.) Let $F(t)$ be the number of followers where F_0 is the initial value. Use the information above to find the function for $F(t)$. Define t.

$F(t) = 75 \cdot 3^{t/8}$

t = number of months (since appearance on Today Show)

Recall, that this would be equivalent to $F(t) = 75e^{k \cdot t}$ (for some constant k) using base e.

$F(8) = 75e^{0.1373(8)}$

expl 1 continued:

b.) So, we know $F(t) = 75e^{k \cdot t}$ and, at the same time, $F(t) = 75 \cdot 3^{1/8}$. So, doesn't that mean that $75e^{k \cdot t} = 75 \cdot 3^{1/8}$. Can you solve for that k? Once we know that, we can write this function in terms of base e. Round to four decimal places.

$\frac{75e^{kt}}{75} = \frac{75 \cdot 3^{t/8}}{75}$

$e^{kt} = 3^{t/8}$

$\ln e^{kt} = \ln 3^{t/8}$

$kt \cdot \ln e^{1} = \frac{t}{8} \ln 3$

$K = \frac{t}{8} \frac{\ln 3}{t}$

$\Rightarrow k = \frac{\ln 3}{8}$

$k \approx 0.1373$

So, $F(t) = 75e^{0.1373t}$

Since k is buried in the exponent, we'll take the natural log of both sides to force it out of there.

It's fun to see how this is related to Theorem 10 from the previous section.

c.) Recall that this k had meaning when we last saw it. What is the *monthly* rate of follower growth in percent form?

$K = 0.1373$

This is a benefit of knowing k.

The monthly rate of growth of Tonya's followers is 13.73% per month.

So far, we've been doing algebra. The calculus comes in when we start asking, "How fast is this number of followers changing?" So, what is the rate of change, or derivative of this function?

Theorem 12

Recall $\frac{d}{dx} e^x = e^x \ln e^1$

$$\frac{d}{dx} a^x = (\ln a) a^x$$

We start here but our exponent is a function of the independent variable, *not* just x. So, we need the Chain Rule too.

Chain Rule for Exponentials with Base a:

$$\frac{d}{dx}\left(a^{g(x)}\right) = (\ln a) \cdot a^{g(x)} \cdot g'(x)$$

As the Chain Rule has done before, this is the original rule with the derivative of the function $g(x)$ tacked on at the end.

$\frac{d}{dx}(a^{g(x)}) = (\ln a)(a^{g(x)}) \cdot g'(x)$ (pg 2)

$F(t) = 75 \cdot 3^{t/8}$ (from #1a)

expl 1 continued:

d.) How fast is the number of followers changing after 20 months? (We'll assume the growth does *not* slow down.) Include units.

$g(t) = t/8 = \frac{1}{8}t \qquad a = 3$

We know $F(t)$. Now find $F'(t)$.

$F'(t) = 75(\ln 3)\, 3^{t/8} \cdot \frac{1}{8}$ — constant multiple rule (1.5)

$F'(t) = \frac{75}{2}(\ln 3) \cdot 3^{t/8}$ — $g(t) = \frac{1}{8}t$, $g'(t) = \frac{1}{8}$

$F'(20) = \frac{75}{8}(\ln 3) \cdot 3^{20/8} \approx 161$ followers/month

e.) As a sidenote, using the function in the form $F(t) = 75 \cdot 3^{1/8}$ can be useful if you want to find certain values *without* the calculator. What happens when you are asked to find the number of followers for some number of months that is a multiple of 8? Choose any such number of months and find and interpret the function value. (If you choose too big of a value for t, you might be tempted to use the calculator. Do *not* feel bad about this. Relish in the fact that you can and just do it.)

$t = 24$ months

$F(t) = 75 \cdot 3^{t/8}$

$F(24) = 75 \cdot 3^{24/8}$

$= 75 \cdot 3^{3}$

$= 2025$ followers

Derivatives of Log, Base *a*, Functions:

Theorem 13

$$\frac{d}{dx}\log_a x = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}$$

Again, if you are taking the log of "a function in x", *not* just x, we have

$$\frac{d}{dx}\left(\log_a g(x)\right) = \frac{g'(x)}{g(x) \cdot \ln a}.$$

Oh, blessed art thou Chain Rule.

$\frac{1}{g(x) \cdot \ln a} \cdot g'(x)$

expl 2: Find the following derivatives. Simplify when it makes sense.

a.) $y = 15^x$

$\frac{dy}{dx} = y' = (\ln 15)\, 15^x$

Practice notation.

$\frac{dy}{dx}$ or y' or $\frac{d}{dx}(R)$

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$\frac{d}{dx}(a^x) = (\ln a)\, a^x$

b.) $y = \log_5(6x^2 + e^x)$

$\frac{dy}{dx} = \frac{12x + e^x}{(6x^2+e^x)\cdot \ln 5}$

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$\frac{d}{dx}(\log_a g(x)) = \frac{g'(x)}{g(x)\cdot \ln a}$

c.) $R(x) = 9^x \cdot \log_{12} x$

$\frac{d}{dx}(fg) = f'g + fg'$

What seeth thou, young personage, thy Product Rule?

$\frac{dR}{dx} = \underbrace{(\ln 9)\cdot 9^x}_{f'} \cdot \underbrace{\log_{12} x}_{g} + \underbrace{9^x}_{f} \cdot \underbrace{\frac{1}{x \ln 12}}_{g'}$

$= (\ln 9) 9^x \cdot \log_{12} x + \frac{9^x}{x \ln 12}$

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$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

d.) $D = 4 \cdot 3^{5w}$

D' or $\frac{dD}{dw} = 4 \cdot (\ln 3)\, 3^{5w} \cdot 5$ ← $g'(w) = 5$

$\frac{dD}{dw} = 20 \cdot (\ln 3)\, 3^{5w}$

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$\frac{d}{dx}(a^{g(x)}) = (\ln a) a^{g(x)} \cdot g'(x)$

expl 3: Find the second derivative of $D = 4 \cdot 3^{5w}$. Call it $\frac{d^2D}{dw^2}$.

$\frac{d^2D}{dw^2} = D'' = \underbrace{20(\ln 3)}_{(1.5)\text{ constant multiple rule}} \cdot (\ln 3)\, 3^{5w} \cdot 5$

$\frac{d^2D}{dw^2} = 100 \cdot (\ln 3)^2 \cdot 3^{5w}$

$\frac{d}{dw}(e^{5w}) = e^{5w} \cdot 5$

$= 5e^{5w}$

Does $3^{5w} \cdot 5 = 15^{5w}$?

No, there's no such exponent rule.

$a^n \cdot b^m = (a \cdot b)^{n \cdot m}$