We turn from local extrema to absolute extrema.

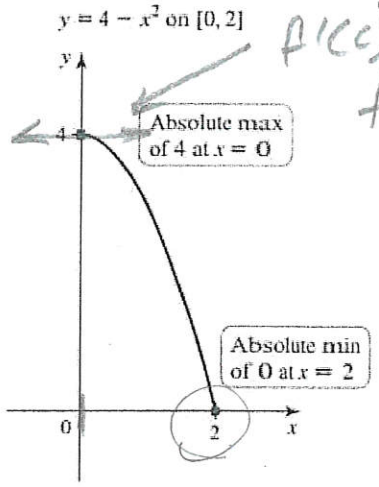
Business Calculus
Class notes
Section 3.4: Applications of Derivatives: Optimization (Absolute Maximums and Minimums)

We might have a function that describes the revenue of a company and be interested in maximizing that revenue. Say you know the trajectory of a near-Earth object and want to know its minimum distance to our planet. We start by defining absolute (a.k.a. global) maximums and minimums as seen back in algebra. The set D is the entire **domain** of the function.

DEFINITION Absolute Maximum and Minimum

Let f be defined on a set D containing c. If $f(c) \geq f(x)$ for every x in D, then $f(c)$ is an **absolute maximum** value of f on D. If $f(c) \leq f(x)$ for every x in D, then $f(c)$ is an **absolute minimum** value of f on D. An **absolute extreme value** is either an absolute maximum value or an absolute minimum value.

The graph shown here has both an absolute max and absolute min. Do you see why the definition above describes the points?



Singular forms: Minimum, Maximum, Extremum

Plural forms: Minima, Maxima, Extrema

If we are asked, "Where does the extremum occur?", we give the x-value or $x = c$.

If we are asked, "What is the extremum?", we give the y-value or $f(c)$.

We often state this as, "The extremum is _____ at _____."
(y-value) (x-value)

We have explored how these points *may* be where $f'(c) = 0$. You should see this for the absolute maximum above but *not* the minimum. (The minimum, in fact, occurs at the endpoint $x = 2$ but $f'(2) \neq 0$.) We have much more to investigate.

Finding Absolute Extrema Over Closed Intervals:

If we have a function and are asked to find its extrema, it's nice to know when it is guaranteed to exist. We have this piece of the puzzle.

> **THEOREM 4.1 Extreme Value Theorem**
> A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

Recall: Definition: Closed Interval: A closed interval is one where the endpoints are included in the interval such as $[0, 1]$ or $[a, b]$.

Recall: Definition: Open Interval: An open interval is one where the endpoints are *not* included in the interval such as $(0, 2)$ or (a, b).

An interval can also be half-open (half-closed).

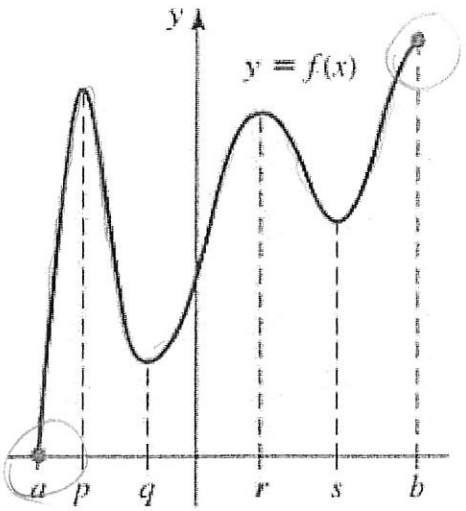
expl 1: The function pictured here is defined on the closed interval $[a, b]$.

a.) Is the function continuous on $[a, b]$?

yes.

b.) Does the Extreme Value Theorem (EVT) apply to this function? If so, find the absolute max and min. Where do the extrema occur?

Yes. The EVT applies.



The abs max is f(b) at x=b.

The abs min is f(a) at x=a.

We will use the procedure on the following page. It is called Theorem 8 in our book.

Theorem 8: Procedure for Finding Absolute Extrema Over a Closed Interval:

Maximum–Minimum Principle 1

Suppose f is a continuous function defined over a closed interval $[a, b]$. To find the absolute maximum and minimum values over $[a, b]$:

We must be given a closed interval.

a. Find $f'(x)$.

b. Determine all critical values in $[a, b]$. That is, find all c in $[a, b]$ for which

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

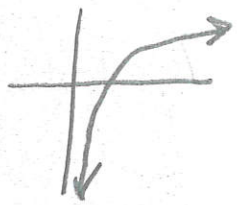
c. List the values from step (b) along with the endpoints of the interval:

$$a,\ c_1,\ c_2,\ \ldots,\ c_n,\ b.$$

d. Evaluate $f(x)$ for each value in step (c):

$$f(a),\ f(c_1),\ f(c_2),\ \ldots,\ f(c_n),\ f(b).$$

$y = \ln x$



Domain: $(0, \infty)$

The largest of these is the **absolute maximum of f** over $[a, b]$. The smallest of these is the **absolute minimum of f** over $[a, b]$.

from 2.3: $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$

expl 2: Conduct this procedure to find any absolute extrema over the interval [-1, 2].

$y = \ln(x^3 - 2x + 8)$

$y' = \frac{3x^2 - 2}{x^3 - 2x + 8}$ ← y' cannot be undefined because $x^3 - 2x + 8$ cannot be zero.

Since we know $x^3 - 2x + 8 > 0$ (why?), we do *not* have to look for where $f'(x)$ is undefined. Do you see why?

because $x^3 - 2x + 8 \neq 0$

Find where $y' = 0$:

$0 = 3x^2 - 2$

$2 = 3x^2$

$\frac{2}{3} = x^2$

$x = \pm\sqrt{\frac{2}{3}}$ ← critical values for step b.

Continue on next page.

$y = \ln(x^3 - 2x + 8)$

Use the **Value** function in the calculator.

More room for example 2:

x	-1	$-\sqrt{\frac{2}{3}}$	$+\sqrt{\frac{2}{3}}$	2
y	2.197	2.207	1.933	2.485

The abs min is 1.933 at $x = \sqrt{2/3}$.

The abs max is 2.485 at $x = 2$.

What are the extrema? Where do they occur?

expl 3: Conduct this procedure to find any absolute extrema over the interval [-4, 5].

closed Interval → Max-Min Principle!

$h(x) = (x+3)^{2/3} - 5$

$h'(x) = \frac{2}{3}(x+3)^{-1/3} \cdot 1$

$h'(x) = \frac{2}{3(x+3)^{1/3}}$ ← $h'(x)$ will never be 0 because the top will never be zero.

$h'(x)$ is undefined where bottom = 0 (or at $x = -3$)

So, critical value is $x = -3$.

x	-4	-3	5
$h(x)$	-4	-5	-1

The abs min is -5 at $x = -3$.

The abs max is -1 at $x = 5$.

Connection Between Local and Absolute Extrema:

If the conditions hold for the principle below, then we may be able to shortcut the process (a bit). Notice this does *not* require a closed interval. However, this does require the second derivative is *not* zero. If it is zero, we must fall back to the First Derivative Test.

Maximum–Minimum Principle 2

Suppose f is a function such that $f'(x)$ exists for every x in an interval I and there is *exactly one* (critical) value c in I, for which $f'(c) = 0$. Then

$f(c)$ is the absolute maximum value over I if $f''(c) < 0$

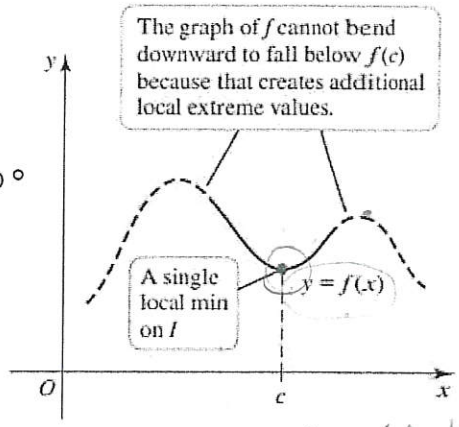
or

$f(c)$ is the absolute minimum value over I if $f''(c) > 0$.

The Second Derivative Test gets us that $f(c)$ is a local extremum. This goes further if an interval contains *only one* such value of c.

Here is a good example of why this theorem must be true. Would you agree with it?

Check the conditions listed above.



not closed interval

expl 4: Find the absolute extremum for the following function on the interval $(-\infty, \infty)$. Max-Min Prin 2

$y = 2xe^{-6x}$

$\frac{d}{dx}(fg) = f'g + fg'$

$y' = 2e^{-6x} + 2x(-6)e^{-6x}$

$y' = 2e^{-6x} - 12xe^{-6x}$

$y' = \frac{2}{e^{6x}} - \frac{12x}{e^{6x}}$

We *cannot* use the first principle as this interval is *not* closed.

$y = e^x$

Dom: $(-\infty, \infty)$

$\frac{d}{dx}(e^{g(x)}) = g'(x)\,e^{g(x)}$

will find c such that $f'(c) = 0$

Continue on next page.

More room for example 4:

$$y' = \frac{2}{e^{6x}} - \frac{12x}{e^{6x}}$$

$$y' = \frac{1}{e^{6x}}(2-12x)$$

$$0 = \frac{1}{e^{6x}}(2-12x)$$

$$2-12x = 0$$

$$12x = 2$$

$$x = 1/6$$

This is exactly one crit. value where $f'(c) = 0$. (max-min principle 2)

Need y'': $\frac{d}{dx}(fg) = f'g + fg'$

$$y' = 2e^{-6x} - (12x)e^{-6x}$$

$$y'' = -12e^{-6x} - (12e^{-6x} + 12x(-6)e^{-6x})$$

$$y'' = -12e^{-6x} - 12e^{-6x} + 72xe^{-6x}$$

$$y'' = -24e^{-6x} + 72xe^{-6x}$$

$$y'' = \frac{-24}{e^{6x}} + \frac{72x}{e^{6x}}$$

Go through the principle bit by bit. Then use the formula for y to find the extremum itself.

$$y''\Big|_{x=1/6} = \frac{-24}{e^1} + \frac{72(1/6)}{e^1} \approx -4.4$$

y evaluated at $x = 1/6$

$y = 2xe^{-6x}$

$y = 2(1/6)e^{-1} \approx 0.123$

We see that this is less than 0.

So, Max-Min Principle 2 says there's an abs max of 0.123 at $x = 1/6$.