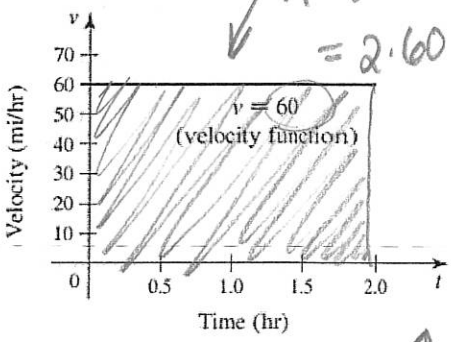
The area under a function has special meaning. Here, we explore it and link it back to our idea of integrals.

Business Calculus
Class notes
Section 4.2: Antiderivatives as Areas and Riemann Sums

So, we know that $f'(x)$ is the rate of change of a function shown as the slope of the tangent line at some x-value. That was pretty cool but it gets even cooler when we investigate the graphical meaning of $\int f(x)dx$.

For our first exploration, consider a car moving along a straight road at a velocity of 60 mph. This is a constant velocity so its graph is a horizontal line, isn't it? Here's a picture of the first two hours of the drive.



$A = l \cdot w$
$= 2 \cdot 60 = 120$ miles

Recall $s'(t) = v(t)$

How far does it travel in this time? (Distance would be $s(t) = \int v(t)dt$.)

$s(t) = \int 60\,dt$

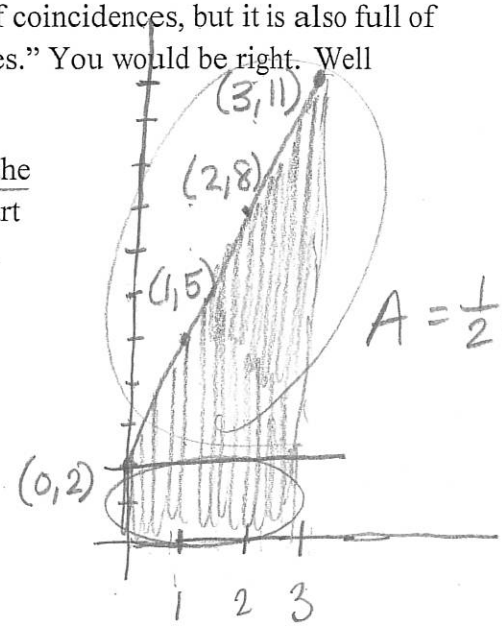
We can call the distance traveled the car's **displacement**. Now, and this is the truly mind-blowing part, draw in the right side of the rectangle this function makes with the axes and find the area of this rectangle.

How is this area related to the total distance driven?

distance = rate · time
= 60 mph · 2 hr
= 120 miles

Do you think it is a coincidence that this area equals the total distance driven? Life is full of coincidences, after all. "No", you say. You say, "Life is full of coincidences, but it is also full of beautiful math that explains at least some of those coincidences." You would be right. Well done. Let's get more creative with this velocity function.

example 1: Let $v(t) = 3t + 2$ miles per hour at t hours. Find the displacement (total distance traveled) from $t = 0$ to $t = 3$. Start with a quick graph and shade the area indicated. Include units.



$A = \frac{1}{2}bh$

Because $v(t) = s'(t)$, we know

$s(t) = \int v(t)\,dt$

$s(t) = \int (3t+2)\,dt$

Area $= 2 \cdot 3 + \frac{1}{2} \cdot 3 \cdot 9$

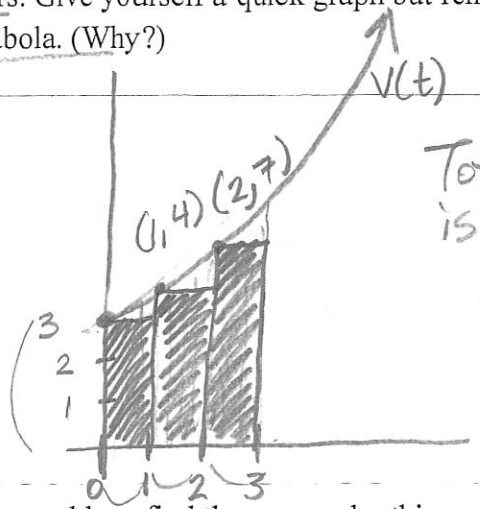
$= 6 + 13.5 = 19.5$ miles

$v(2) = 2^2 + 3 = 7$

$v(1) = 1^2 + 3 = 4$

Not All Functions Are Straight Lines:

Let's say our car's velocity function is $v(t) = t^2 + 3$ miles per hour where t is the number of hours. Give yourself a quick graph but remember that we will only need the right side of the parabola. (Why?)



Total distance is this area.

Label the t-values 0, 1, 2, and 3. Consider the *distance driven* in these first three hours.

How would we find the area under this curve for [0, 3]? It's all curvy on top so it's *not* so easy as before. However, we could *approximate* the area by using a series of rectangles.

Consider your drawing of $v(t) = t^2 + 3, t \geq 0$. From the points at $t = 0$, 1, and 2, draw horizontal lines across until you run into the next t-value. This forms rectangles which we shade. Calculate this shaded area.

The shaded area is about $3 + 4 + 7 = 14$ miles

	$t = 0$	$t = 1$	$t = 2$
Area of rectangle	$1 \cdot 3 = 3$	$1 \cdot 4 = 4$	$1 \cdot 7 = 7$

How well do these rectangles approximate the actual area? Did we overestimate or underestimate the actual area?

This is called a **regular partition** of the interval [0, 3].

We do this work by dividing the interval [0, 3] into three subintervals, [0, 1], [1, 2], and [2, 3].

Other calculus sources may discuss finding the areas of the rectangles based on the left endpoints of each interval (which we did), the right endpoints, the middle point in each interval, or any point whatsoever in these subintervals.

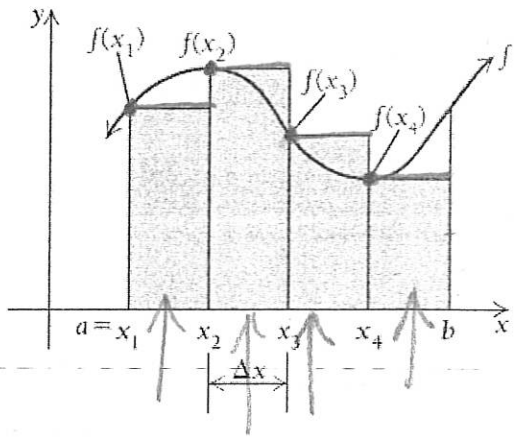
The points you choose are always on the function itself and determine the height (and therefore, area) of each rectangle. The width of each rectangle is determined by how many rectangles you choose to use. You can imagine how the approximate area under the curve would be a bit different depending on which points you use.

Our book, just to show the rough idea, only covers the procedure using the left endpoints.

Riemann Sums:

Wikipedia.org tells me that Riemann Sums are named after the nineteenth century German mathematician **Bernhard Riemann**. One can only assume he was teased mercilessly for his name until he showed them all by becoming a great mathematician. Bet they got nothing named after them.

Here's a picture from our book with a generic function $f(x)$. The interval $[a, b]$ is broken into four subintervals using x_1, x_2, x_3, and x_4.



Again, notice how the $f(x)$ values determine the height of each rectangle. Since we use four *equal-sized* subintervals, each rectangle has the same width that we'll call Δx.

Hence, the total area under the curve from $x = a$ to $x = b$ would be estimated by $f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + f(x_4)\cdot\Delta x$.

Summation (Sigma) Notation:

Writing the calculations for these areas is repetitive and hard on the hand. Mathematicians, being the lazy buggers they are, came up with shorthand notation. We will use the Greek capital letter Σ (sigma).

If we want to write $1 + 2 + 3 + \ldots + 10$, we will abbreviate that as $\sum_{k=1}^{10} k$. The **index** k takes on every integer value from 1 through 10. This is an arbitrary symbol (a dummy variable) and you could use n or m as easily as k. The **summand** is the name for the expression that follows the sigma (and is the thing you add again and again as the index changes from the **lower limit** to the **upper limit**). In this case, the summand is k but it could be any expression. We will play with this notation a bit but mainly use it for our Riemann Sums.

expl 1: Write out the following in expanded form. Do *not* bother finding the sum.

$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

Constant Multiple Rule $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k.$

Addition Rule $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$

There are many rules; here are only sum of them.

Back to Riemann Sums:

We will use summation notation to write $f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + f(x_4)\cdot\Delta x$ as simply $\sum_{i=1}^{4} f(x_i)\cdot\Delta x$.

Our Old Friend Limit Lends a Hand:

Imagine if we had used more rectangles. We could have pinpointed the area even better with more, skinnier rectangles. The more rectangles we used, the better our approximation would become.

What if we had an infinite number of rectangles? You say we can't do that? Okay, but we can *imagine* it, right?

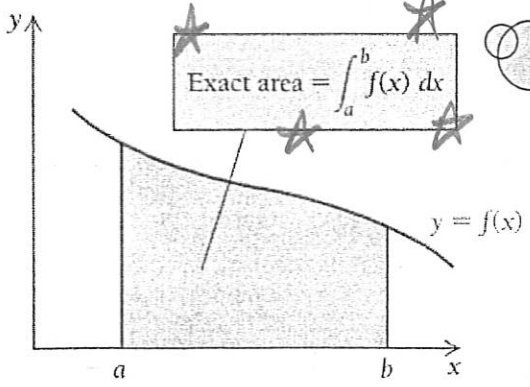
To find this area exactly, let n be the number of rectangles we subdivide the area into. We will take the limit, as $n \to \infty$, of the sum of these rectangles' areas. This, we will see as we go through the chapter, is the *exact* area under the curve. For the example we started with, this would be the *exact* distance the car had traveled.

Now, catch this one. If we let the number of rectangles approach ∞, then the width of each rectangle would approach 0, wouldn't it? Hence, we have this gem from our book which brings back in the integral, but this time we have a *definite* integral.

Let $y = f(x)$ be continuous and nonnegative over an interval $[a, b]$. A **definite integral** is the limit as $\Delta x \to 0$ of the Riemann sum of the areas of rectangles under the graph of $y = f(x)$ over $[a, b]$.

$$\text{Exact area} = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\cdot\Delta x = \int_a^b f(x)\,dx.$$

This is a *definite* integral since it uses endpoints.

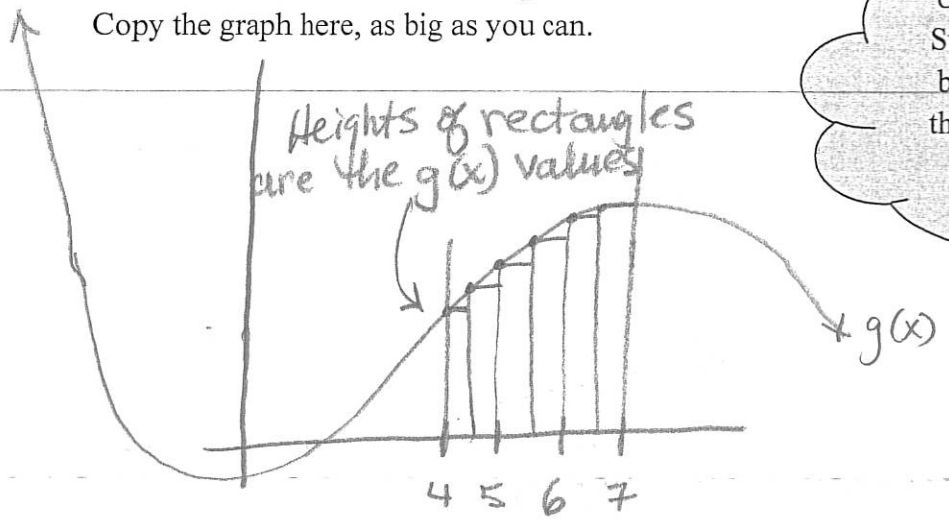


In the next section, we'll learn how to find these areas using definite integrals. For now, the book expects you to find these areas with Riemann Sums when indicated or geometrically (Start with a graph and find the area under the curve with area formulas for rectangles, trapezoids, and triangles.)

$[a,b] = [4,7]$ $n = 6$

example 2a: Approximate the area under the graph of $g(x) = -0.1x^3 + 1.2x^2 + 0.4x - 4.8$ in the interval [4, 7] using six equal-sized subintervals.
We will use the table below to organize ourselves.
Copy the graph here, as big as you can.

Use your calculator to graph. Start on the Standard window but then adjust y-max to see the whole graph. Draw the bit we're interested in.



$\Delta x = \frac{7-4}{6} = \frac{1}{2}$

$\Delta x = \frac{b-a}{n}$

Split the interval [4, 7] into six equal-sized subintervals. We need to figure the x-value and $g(x)$ for each of the left endpoints of these intervals. Here is a table to organize. Use the ~~Value~~ Table function (~~2nd TRACE for Calculate menu~~) on the calculator to find $g(x)$ values. Round to two decimal places when necessary. $+\Delta x$ $+\Delta x$ $+\Delta x$

left endpoint	4	4.5	5	5.5	6	6.5
***g*(*x*)**	9.6	12.19	14.7	17.06	19.2	21.04

Lastly, what is the width of each rectangle? Call it Δx. $\Delta x = 1/2$

Multiply each $g(x)$ value by this width and you have that rectangle's area. Add them up and you get an approximation of the total area under the curve from 4 to 7. Do this now.

In other words, $\sum_{i=1}^{6} g(x_i) \cdot \Delta x$.

$\sum_{1}^{6} g(x_i) \cdot \Delta x$

$= 9.6(\frac{1}{2}) + 12.19(\frac{1}{2}) + 14.7(1/2) + 17.06(\frac{1}{2}) + 19.2(1/2) + 21.04(1/2)$

area $= 46.875$ This estimates the area under $g(x)$ between $x=4$ and $x=7$.

In case you see the term elsewhere, I will point out that these x-values are sometimes referred to as **grid points**. $x_1 = 4, x_2 = 4.5, x_3 = 5, \ldots x_6 = 6.5$.

We can use the calculator for these repetitive calculations.

TI Calculator Instructions:

1. Start on the home screen. Press the **2nd** and then **STAT** buttons to get to the **LIST** command. Arrow over to **MATH**. Select the option **5: sum(**. [This will be the sigma part of the expression.]

2. Navigate back to the **LIST** command as before. This time arrow over to **OPS**. Select the option **5: seq(**. [This will be how you tell it the lower and upper limits for k.]

3. You will enter the expression you want summed using x, the variable x itself, the lower limit of the summation, the upper limit of the summation, and the step or increment (which is Δx), all separated by commas. Finish with two parentheses at the end and hit **ENTER**.

Newer calculators may have a different interface.

expl 2b: Try this process out to check the summation from the previous page. $\sum_{1}^{b} g(x_i) \cdot \frac{1}{2}$

area = 46.89375 Very close to what we got on pg 5.

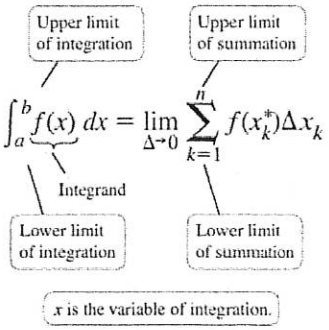
Indefinite Versus Definite Integrals:

It is important to note that the indefinite integrals we saw earlier are very much related but are *not* considered to be areas. An indefinite integral is an antiderivative of a function. In other words, the integral of a function is another function whose derivative gives us the original function.

On the other hand, a definite integral gives us this net area. We will see, in the next section, how the indefinite and definite integrals are related. And, wow, just wow. Seriously, just you wait...

Definite Integral Notation:

It does help to see why the notation is the way it is. The left and right sides of this equation are analogous. (I took it from another book but it's equivalent to what we have above.)



means $\Delta x \to 0$

The sum on the right, as $\Delta \to 0$, becomes a sum with an infinite number of terms, denoted by the integral sign $\int$ on the left. This was designed to look like an elongated S for sum. The **limits of integration**, a and b, also match with $k = 1$ and n from the right.

Recall that the $f(x)$ part of the integral is called the **integrand** and that you should always think of the $\int$ and the dx as two pieces of the same symbol. This dx tells us that the independent variable is x. (A moldering pile of bones, Gottfried Wilhelm Leibniz, invented this notation. Albeit, he was moldering less in 1675 when he introduced the notation.)