Some theorems get all the luck in naming. This one really does live up to its name. We connect antiderivatives and definite integrals.

Business Calculus
Class notes
Section 4.3: The Fundamental Theorem of Calculus: Area and Definite Integrals

The book has a lovely proof of this FUN-damental theorem. We will simply state it here and spend our time practicing its use.

The Fundamental Theorem of Integral Calculus

If a continuous function f has an antiderivative F over $[a, b]$, where n is the number of subdivisions of $[a, b]$, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx = F(b) - F(a).$$

This part on the left, as you'll remember, is the area under the curve. For our purposes, this gives us a way to find the definite integral that we were exposed to in the last section.

Shorthand notation for $F(b) - F(a)$ is $F(x)\Big|_a^b$.

So, in short, to find the value of a definite integral, we find the integral (as we saw it before) as an antiderivative. We then evaluate the antiderivative at the limits of integration and subtract (top minus bottom).

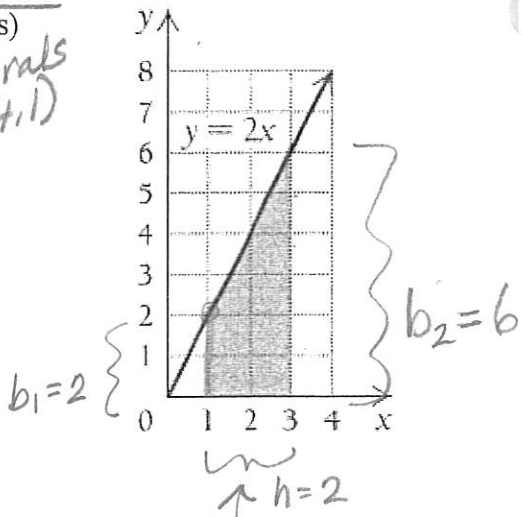
Voilà! We have the area under the function's curve, bounded by the x-axis.

We have come full circle back to the antiderivatives where we began.

Let's practice. I will refer to the theorem as FTC.

$\int c f(x)\,dx = c\int f(x)\,dx$ $\quad\quad$ $\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$

example 1: For the function $f(x) = 2x$ shown to the right, use an integral to find the area under the curve (but above the x-axis) over the interval [1, 3].



area $= \int_1^3 2x^1\,dx$ Indefinite Integrals Rules (4.1)

$= \frac{2x^2}{2} = x^2\Big|_1^3$ F.T.C.

$= 3^2 - 1^2$

$= 8$

As an extra exercise, you could check yourself by using the area formula of a trapezoid. Can you show that it is equal to what we got above?

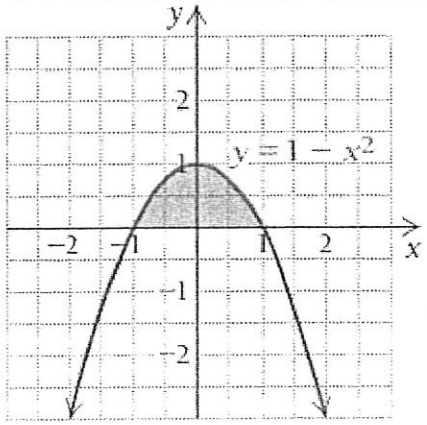
$A = \frac{1}{2}h(b_1 + b_2)$

$= \frac{1}{2}\cdot 2\cdot(2+6)$

$= 8$ ✓

The area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$ where h is the distance between parallel bases b_1 and b_2.

example 2: For the function $f(x) = 1 - x^2$ shown to the right, use an integral to find the area under the curve (but above the x-axis) over the interval [-1, 1].



area $= \int_{-1}^{1} (1 - x^2)\,dx$

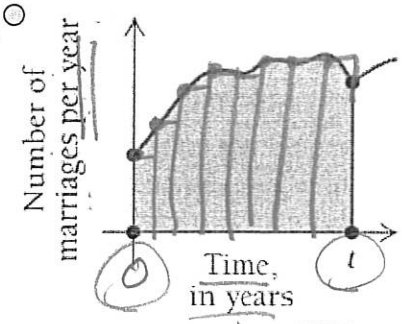
$= \int_{-1}^{1} 1\,dx - \int_{-1}^{1} x^2\,dx$ (4.1 Rules)

$= x - \frac{x^3}{3}\Big|_{-1}^{1}$

$= 1 - \frac{1^3}{3} - \left(-1 - \frac{(-1)^3}{3}\right) = 1 - \frac{1}{3} + 1 + \frac{-1}{3} = \frac{4}{3}$

example 3: Explain what the shaded region represents.

Imagine finding this area with rectangles. Each rectangle has a height (determined by the function) with units of number of marriages per year and a width with units of years.



area = $\frac{\text{number of marriages}}{\text{year}} * \frac{\text{year}}{1}$ = number of marriages

This area is the total number of marriages from $t=0$ to $t=t$ years.

example 4: Use FTC to find the following.

We are finding the area bounded by the t-axis and $f(t)=3t^2+2t$ between the t-values of 0 and 4.

$$\int_0^4 (3t^2+2t)\,dt = 3\int_0^4 t^2\,dt + 2\int_0^4 t\,dt$$

4.1 Rules

$$= \frac{3\cdot t^3}{3} + \frac{2\cdot t^2}{2}$$ 4.1 Rules

$$= (t^3+t^2)\Big|_0^4 = 4^3+4^2-(0^3+0^2)$$

FTC $= 64+16 = 80$

example 5: Use FTC to find the following. (Here, a and b are constants.)

$$\int_a^b e^{2x}\,dx =$$

My a-value is 2 (exponent)

$$= \frac{1}{2}e^{2x}\Big|_a^b$$

From 4.1

$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \quad a \neq 0$ ★

$$= \frac{1}{2}e^{2b} - \frac{1}{2}e^{2a}$$

OR

$$= \frac{1}{2}(e^{2b} - e^{2a})$$

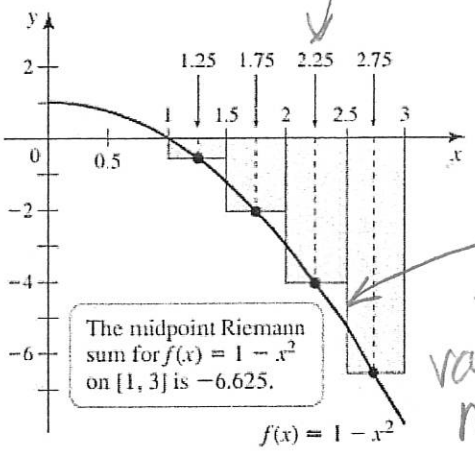
When Functions Dip Below the x-axis:

We now have that definite integrals find the area under a curve. Technically, the area we found was bound by the x-axis, wasn't it? We specifically kept to examples where the function was drawn above the x-axis. In other words, our functions were non-negative. Here, we see what it looks like when they're *not*. We'll consider it from the perspective of Riemann Sums to better see how we get our answer.

Take a look at this picture of $f(x)=1-x^2$ on the interval $[1, 3]$.

Since these function values are negative, the formula $\sum_{k=1}^{n} f\left(x_k^*\right)\Delta x$ would necessarily sum up negative numbers and end up with a negative number.

But area is *not* negative! Or, is it?



f(x) values are negative here

The Riemann Sum being negative does *not* mean it's useless. We just have to reimagine what it does mean. We are getting the *negative* of the area bounded by the function and the x-axis. That could be useful. Here's a definition to start making sense of this.

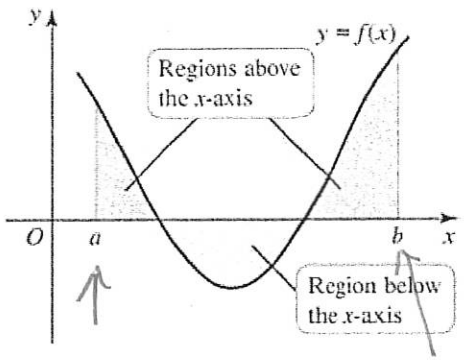
DEFINITION **Net Area**

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

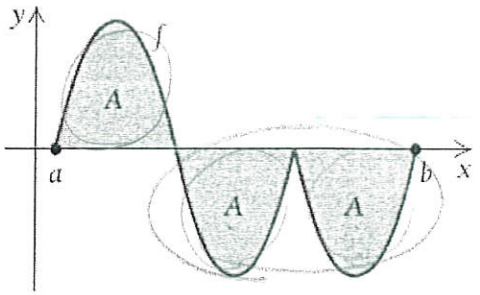
Here's a picture of a function which is both below and above the x-axis.

We will count area above as *positive* and area below as *negative*.

The Riemann Sum or definite integral will naturally produce those signs.



example 6: Here, the areas of the shaded regions are given. Would the definite integral $\int_a^b f(x)dx$ be positive, negative, or zero? What value would you give this integral?



$\int_a^b f(x)dx$

$= A - 2A = -A$

above x-axis, below x-axis

example 7: A computer motherboard company has found that their marginal profit, in dollars per motherboard, is $P'(x) = 2.6x^{0.1}$ where x is the number of motherboards produced and sold.

a.) Find the company's profit for selling 1200 motherboards. (~~Assume the profit for selling 0 motherboards is \$0.~~)

Recall, $P(x) = \int P'(x)dx$. Knowing $P(0)$ lets us solve for C.

$P(x) = \int P'(x)dx$

$P(1200) = \int_0^{1200} 2.6x^{0.1}\,dx$ ↓ 4.1 Constant Multiple Rule

$= 2.6\int_0^{1200} x^{0.1}\,dx$

$= \frac{2.6x^{1.1}}{1.1}\Big|_0^{1200} \approx$ \$5763.42 profit when we sell 1200 boards

b.) A customer has increased their order from 1200 to 1500 motherboards. Find the additional profit this yields for our motherboard company. Write it as a definite integral.

$P(1500) - P(1200) = \int_{1200}^{1500} 2.6x^{0.1}\,dx$

$= \frac{2.6}{1.1}x^{1.1}\Big|_{1200}^{1500} = \frac{2.6}{1.1}(1500)^{1.1} - \frac{2.6}{1.1}(1200)^{1.1}$

$\approx$ \$1603.42 ✓