

The first and second derivatives of a function can tell you a lot about its graph.

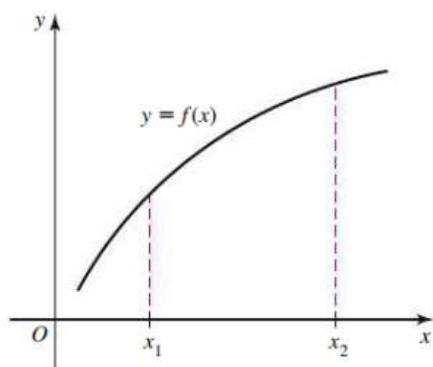
Calculus I  
Class notes

The First and Second Derivative Tests (section 4.3)

The concepts of increasing and decreasing as well as concavity come back from algebra here. We will understand how the first and second derivatives help us determine these important characteristics of a function's graph. First, let's review.

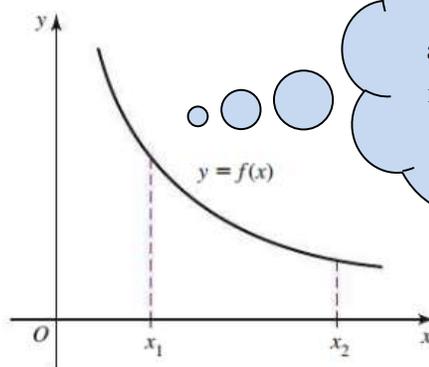
**DEFINITION Increasing and Decreasing Functions**

Suppose a function  $f$  is defined on an interval  $I$ . We say that  $f$  is **increasing** on  $I$  if  $f(x_2) > f(x_1)$  whenever  $x_1$  and  $x_2$  are in  $I$  and  $x_2 > x_1$ . We say that  $f$  is **decreasing** on  $I$  if  $f(x_2) < f(x_1)$  whenever  $x_1$  and  $x_2$  are in  $I$  and  $x_2 > x_1$ .



$f$  increasing:  $f(x_2) > f(x_1)$   
whenever  $x_2 > x_1$

(a)



$f$  decreasing:  $f(x_2) < f(x_1)$   
whenever  $x_2 > x_1$

(b)

An increasing function will rise as you travel left to right. A decreasing function will fall.

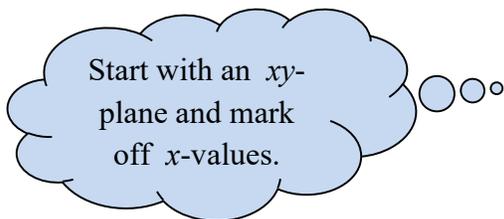
As you look at these graphs, imagine some tangent lines. Go crazy and draw a few in. What do you notice about those of an increasing graph versus those tangent lines on the decreasing graph?

This realization brings us to our first theorem of this section. Once again, the proof is given in the book but your examples should be pretty convincing.

**THEOREM 4.7 Test for Intervals of Increase and Decrease**

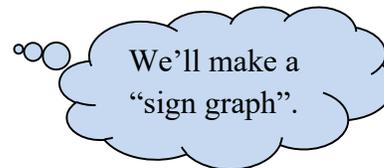
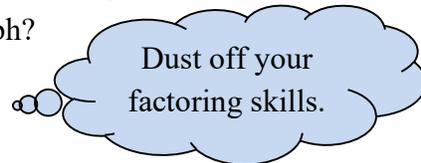
Suppose  $f$  is continuous on an interval  $I$  and differentiable at all interior points of  $I$ . If  $f'(x) > 0$  at all interior points of  $I$ , then  $f$  is increasing on  $I$ . If  $f'(x) < 0$  at all interior points of  $I$ , then  $f$  is decreasing on  $I$ .

expl 1: Sketch a graph of a function  $f(x)$  such that  $f'(x) < 0$  on  $(-2, 1)$  and  $(4, \infty)$  but  $f'(x) > 0$  on  $(-\infty, -2)$  and  $(1, 4)$ .



On homework, you will see these questions as multiple-choice. What if you were told that  $f'(-2) = 0$ ? What would that tell you about the function's graph?

expl 2: Find intervals on which  $f$  is increasing or decreasing.  
 $f(x) = x^4 - 4x^3 + 4x^2$



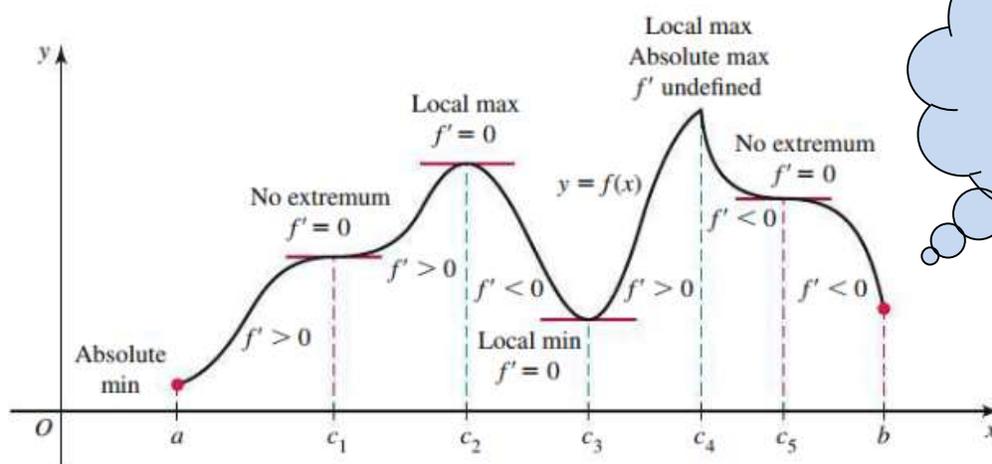
## Sign Graphs:

The process, as seen in the previous example, will be to find critical points (where  $f'(x) = 0$  or is undefined) and then to fill out a **sign graph** using these values as endpoints of intervals. For each interval, we need only find out if  $f'(x)$  is positive or negative. Hence, we only need one value in each interval to test. If one  $f'(x)$  is positive in the interval, then *all*  $f'(x)$  values will be.

You can always graph  $f(x)$  on the calculator to check your conclusions.

## Local Minimums and Maximums:

A critical point *may* be where a local min or max occurs but it does *not* have to be. Check out this graph where we have both  $f'(c) = 0$  and  $f'(c)$  is undefined.



Find the points where  $f'(c) = 0$  but there is *not* a max *nor* min.

Find the local maxes and mins above. Do you notice something interesting happening on either side of those extrema? We have the **First Derivative Test** to put it in words.

### THEOREM 4.8 First Derivative Test

Assume  $f$  is continuous on an interval that contains a critical point  $c$ , and assume  $f$  is differentiable on an interval containing  $c$ , except perhaps at  $c$  itself.

- If  $f'$  changes sign from positive to negative as  $x$  increases through  $c$ , then  $f$  has a **local maximum** at  $c$ .
- If  $f'$  changes sign from negative to positive as  $x$  increases through  $c$ , then  $f$  has a **local minimum** at  $c$ .
- If  $f'$  is positive on both sides near  $c$  or negative on both sides near  $c$ , then  $f$  has no local extreme value at  $c$ .

### Connection Between Local and Absolute Extrema:

This one is useful when interpreting a function's graph.

#### **THEOREM 4.9 One Local Extremum Implies Absolute Extremum**

Suppose  $f$  is continuous on an interval  $I$  that contains exactly one local extremum at  $c$ .

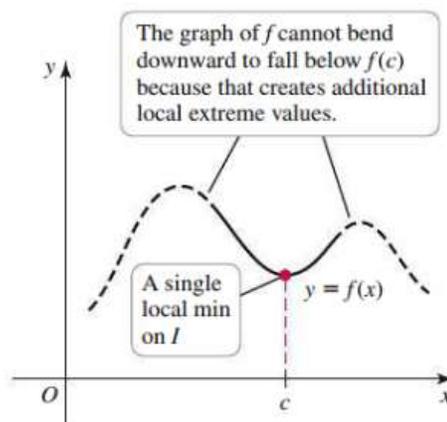
- If a local maximum occurs at  $c$ , then  $f(c)$  is the absolute maximum of  $f$  on  $I$ .
- If a local minimum occurs at  $c$ , then  $f(c)$  is the absolute minimum of  $f$  on  $I$ .

Here is a good example of why this theorem must be true. Would you agree with it?

expl 3: For the following function, locate the critical points of  $f$  in the interval  $[-2, 4]$ . Use the First Derivative Test to find local extrema. Lastly, check if Theorem 4.9 applies and, if so, use it to identify an absolute extremum.

$$f(x) = 2x^3 + 3x^2 - 12x + 1$$

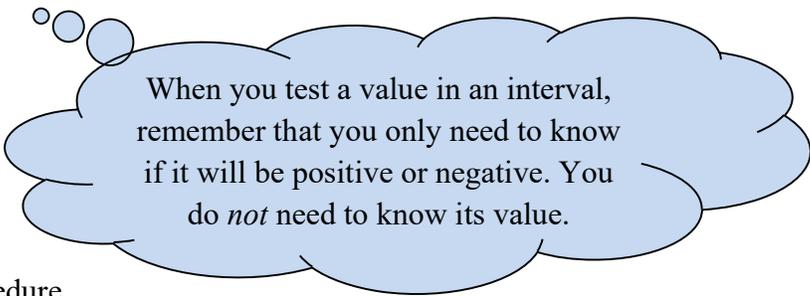
We'll make a "sign graph".



More room on next page.

Do you recall why we say a local max does *not* occur at  $x = -2$ ?

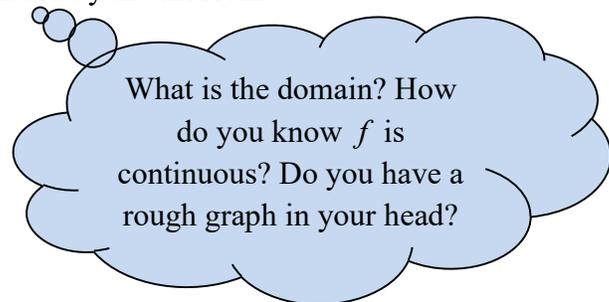
expl 3 continued: Theorem 4.9 will give us one absolute extremum. However, you will need some more work to find the other. Recall the procedure for finding absolute extrema on a closed interval from earlier.



Graph on the window  $[-10, 10] \times [-10, 150]$  to check. Remember we are interested in the interval  $[-2, 4]$ .

expl 4: Verify that this function satisfies the conditions of Theorem 4.9 on its domain. Then find the location and value of the absolute extremum guaranteed by the theorem.

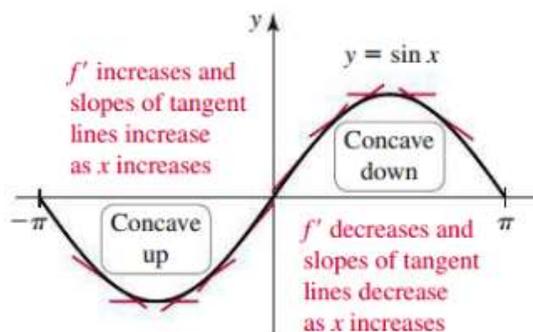
$$h(x) = 4x^2 - 3x + 4$$



## Concavity and Inflection Points:

Take a look at this graph. Algebra taught us the concepts of concave up and down. Now, we apply calculus to this graph.

There is a point in the middle where the graph changes from concave up to concave down. This point is an **inflection point**. We will also see that the slopes of these tangent lines stop increasing and start decreasing.



Below, we see how the derivative plays a role.

### DEFINITION Concavity and Inflection Point

Let  $f$  be differentiable on an open interval  $I$ . If  $f'$  is increasing on  $I$ , then  $f$  is **concave up** on  $I$ . If  $f'$  is decreasing on  $I$ , then  $f$  is **concave down** on  $I$ .

If  $f$  is continuous at  $c$  and  $f$  changes concavity at  $c$  (from up to down, or vice versa), then  $f$  has an **inflection point** at  $c$ .

Now, this mentions that "If  $f'$  is increasing [or decreasing] on  $I \dots$ "

How does calculus tell us when a function is increasing or decreasing? That, you say, involves finding the function's derivative. What is the derivative of  $f'$ ? And so, we have this test for concavity.

### THEOREM 4.10 Test for Concavity

Suppose  $f''$  exists on an open interval  $I$ .

- If  $f'' > 0$  on  $I$ , then  $f$  is concave up on  $I$ .
- If  $f'' < 0$  on  $I$ , then  $f$  is concave down on  $I$ .
- If  $c$  is a point of  $I$  at which  $f''$  changes sign at  $c$  (from positive to negative, or vice versa), then  $f$  has an inflection point at  $c$ .

Do *not* read too much into this. It is *not necessarily true* that if  $f''(c) = 0$ , then  $f''$  *must* have changed sign and you would find an inflection point. Can you think of an example where  $f''(c) = 0$  for some  $c$  but  $f$  has *no* inflection point there?

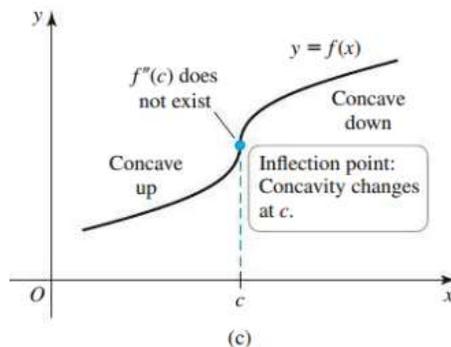
### When the Second Derivative Does *Not* Exist:

An inflection point may also occur when the second derivative does *not* exist. Here is the graph of such a function.

Another example is the function  $f(x) = \sqrt[3]{x} = x^{1/3}$ .

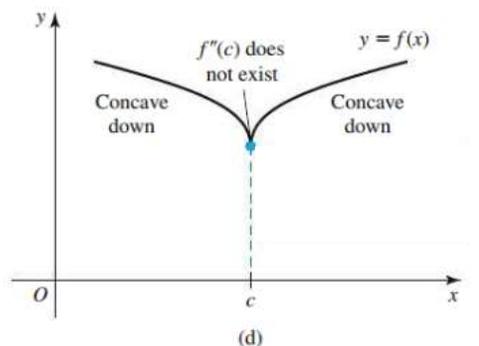
Do you have its graph in your head?

Find the second derivative of  $f(x) = \sqrt[3]{x} = x^{1/3}$  to show it does *not* exist at  $x = 0$  (its inflection point).



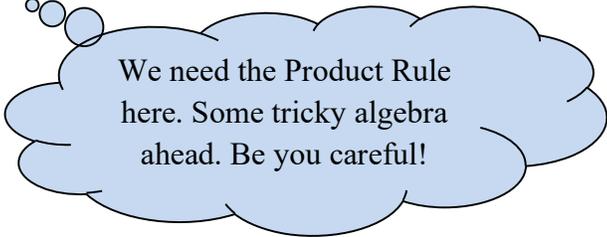
Here's another example of a function whose second derivative does *not* exist.

Here, we do *not* call the point in the middle an inflection point. Do you know why?

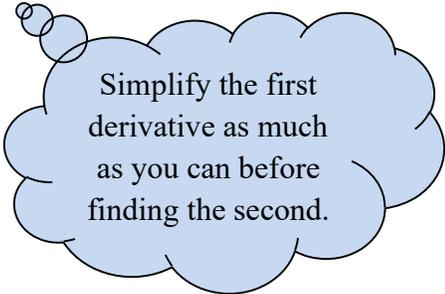


expl 5: Determine intervals on which the following function is concave up and concave down. Identify inflection point(s). Consider the domain interval of  $[-1, \infty)$ .

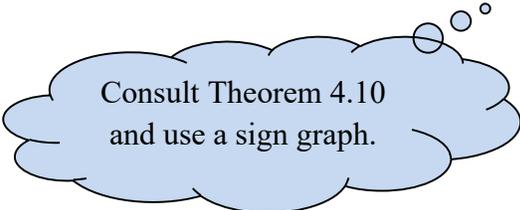
$$g(x) = 8(x+1)^{5/2}(4x-9)$$



We need the Product Rule here. Some tricky algebra ahead. Be you careful!



Simplify the first derivative as much as you can before finding the second.

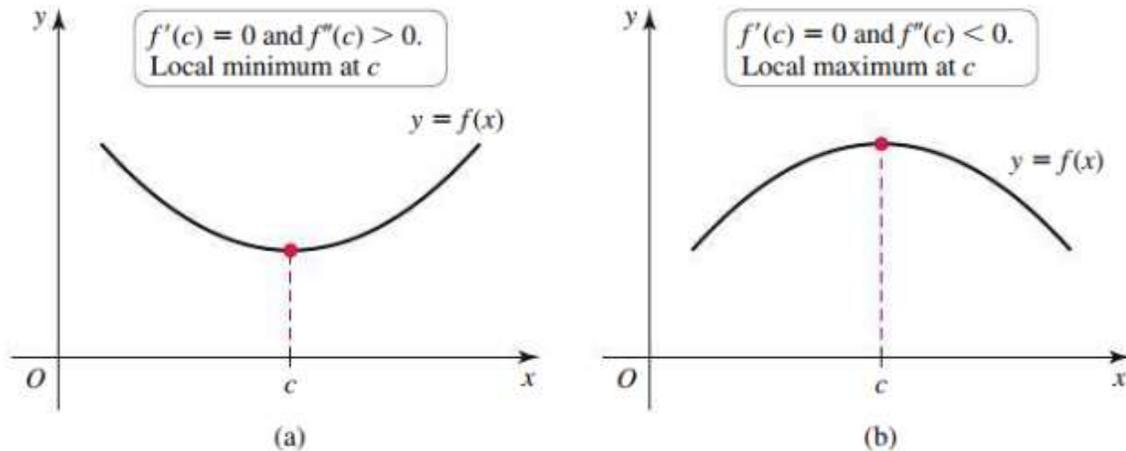


Consult Theorem 4.10 and use a sign graph.

The homework will require exact answers. Graph on  $[-5, 5] \times [-250, 100]$  or similar to check yourself.

### Second Derivative Test:

You may remember from algebra the connection between concavity and local maxes and mins. Take a look at these pictures, noticing the left graph is concave up and the right is concave down.



The Second Derivative Test (proven, in part, with the First Derivative Test) is our last piece of this puzzle.

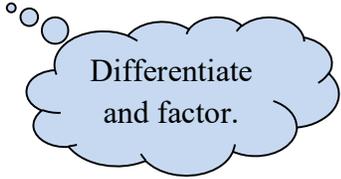
#### **THEOREM 4.11** Second Derivative Test for Local Extrema

Suppose  $f''$  is continuous on an open interval containing  $c$  with  $f'(c) = 0$ .

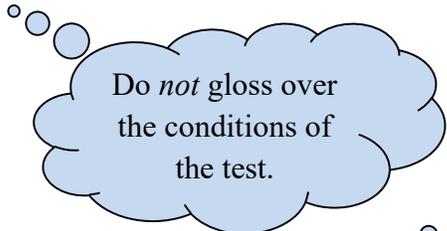
- If  $f''(c) > 0$ , then  $f$  has a local minimum at  $c$  (Figure 4.38a).
- If  $f''(c) < 0$ , then  $f$  has a local maximum at  $c$  (Figure 4.38b).
- If  $f''(c) = 0$ , then the test is inconclusive;  $f$  may have a local maximum, a local minimum, or neither at  $c$ .

expl 6: Find the critical points for the following function. Then use the Second Derivative Test to locate local maxes and mins.

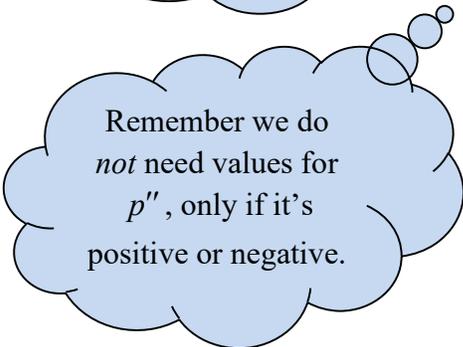
$$p(t) = 2t^3 + 3t^2 - 36t$$



Differentiate  
and factor.



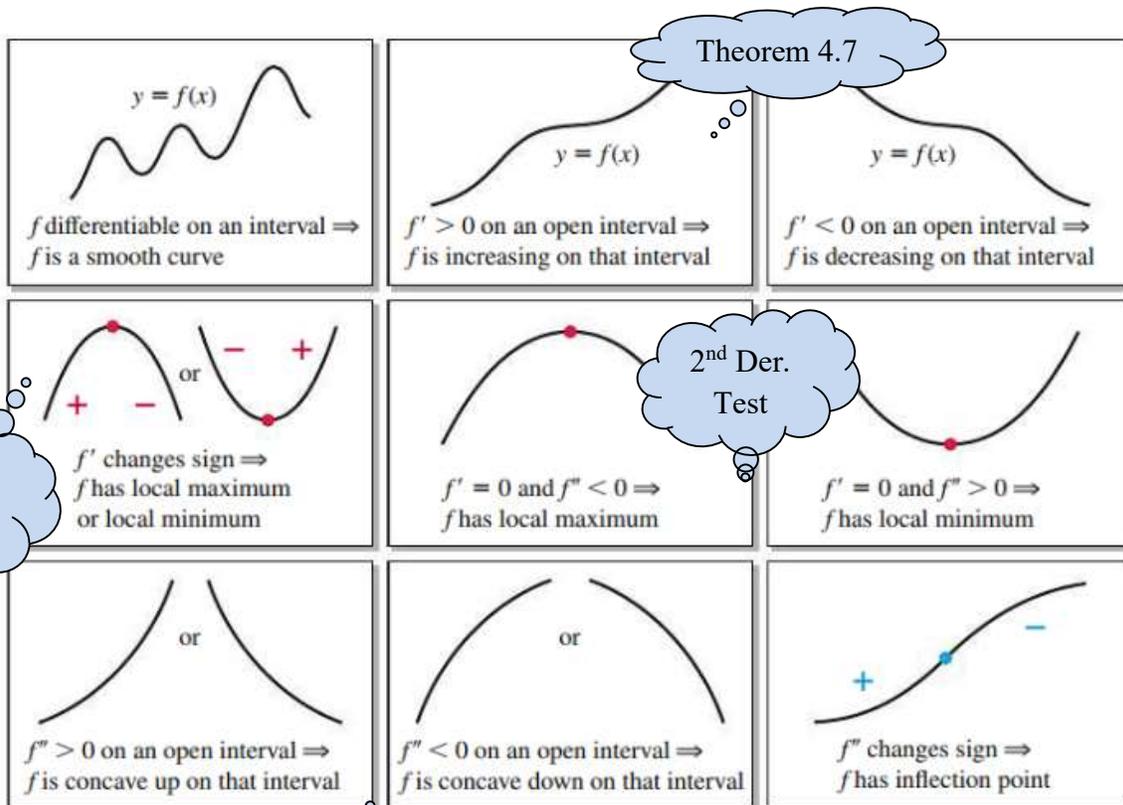
Do *not* gloss over  
the conditions of  
the test.



Remember we do  
*not* need values for  
 $p''$ , only if it's  
positive or negative.

## Recap of Derivative Properties

This section has demonstrated that the first and second derivatives of a function provide valuable information about its graph. The relationships among a function's derivatives and its extreme values and concavity are summarized in **Figure 4.41**.



**Figure 4.41**

bottom row:  
Theorem 4.10