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> We explore the slope of the tangent line on a graph to understand its meaning.

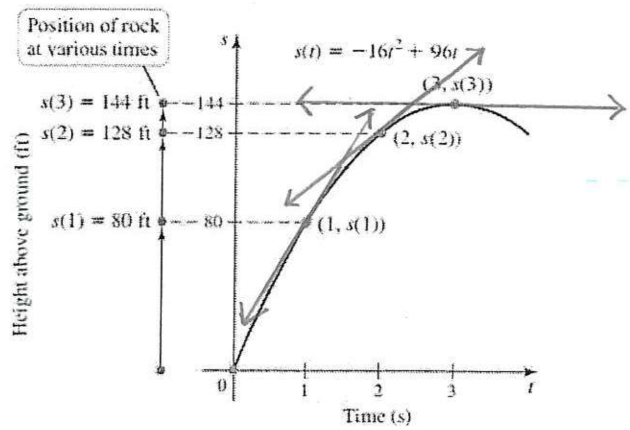
Calculus I
Class notes
Introducing the Derivative (section 3.1)

The slope of a tangent line, which is the **instantaneous rate of change** of the function, takes on different values as you traverse the graph. Recall the graph below we were exploring when we first thought about this.

Recall this is the partial graph for the function $s(t) = -16t^2 + 96t$ where $s(t)$ is the height of a rock t seconds after it has been thrown straight up.

Draw in the three different tangent lines for the values $t = 1, 2,$ and 3.

Notice how the slopes of these lines differ from one another. The slope depends on the value of t because the shape of the graph depends on t.



The values of the slope of a tangent line to the graph (which we found as the limit of the slope of the secant line) comprise a function we call the derivative of $f(x)$. Let's start off with another projectile example.

expl 1: A projectile is fired upward. Its position (height, in feet) above the ground t seconds after release is given by $s(t) = -16t^2 + 100t$. Find the instantaneous rate of change at $t = 1$ second.

$$v_{inst} = \lim_{t \to 1} \frac{s(t) - s(1)}{t - 1}$$

> The fraction here is the slope of the *secant* line between (1, 84) and some other point to its right. Finding the limit of that as t approaches 1, gives us the slope of the *tangent* line at $t = 1$.

$$V_{inst} = \lim_{t \to 1} \frac{-16t^2 + 100t - 84}{t-1}$$

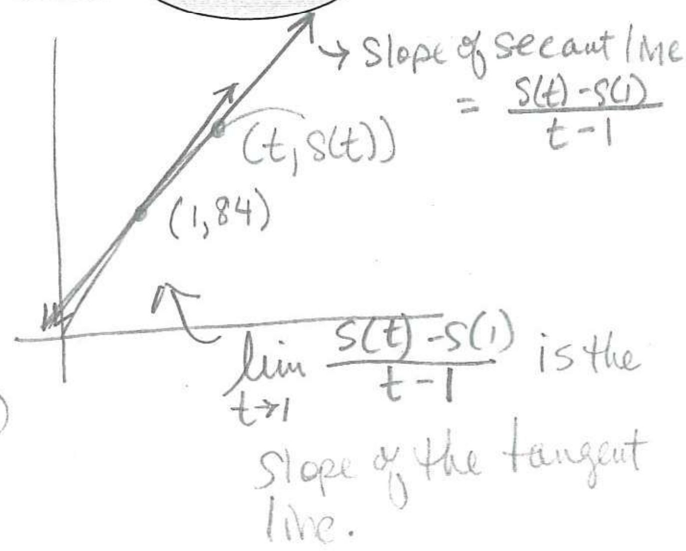
$$= \lim_{t \to 1} \frac{-4(4t^2 - 25t + 21)}{t-1}$$

$$= \lim_{t \to 1} \frac{-4\cancel{(t-1)}(4t-21)}{\cancel{(t-1)}}$$

$$= \lim_{t \to 1} -4(4t-21) = -4(4 \cdot 1 - 21)$$

$$= -4(-17)$$

$$= 68 \text{ ft/sec}$$



$$m = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow m(x_2 - x_1) = y_2 - y_1$$

Interpretation:

Our result means that the projectile's height is *changing* at the rate of 68 feet per second when the projectile has been in the air for 1 second.

This rate of change will be different for another value of t. Later, we will work on finding a formula for this rate of change.

The units we attribute come from the formula for slope. We use units for $s(t)$ divided by units of t.

For now, concentrate again on this point (1, 84) and the fact that we know the slope of the tangent line here is 68. Use good ol' algebra to find the equation of this tangent line that goes through this point. Use *either* starting formula below.

Point-slope Formula	OR	Slope-intercept Formula
$y - y_1 = m(x - x_1)$		$y = mx + b$

$y - 84 = 68(x - 1)$

$y - 84 = 68x - 68$

$y = 68x + 16$

$y = 68x + b$

$84 = 68(1) + b$

$84 = 68 + b$

$16 = b \rightarrow y = 68x + 16$

We start off with a projectile example because it's nice to imagine the rock and how its position varies over time. However, we can do this sort of thing with any function.

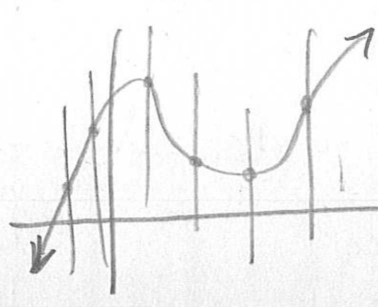
Maybe we have a function that describes a deer population, or radioactive decay of a substance, or the weight of a child. For any function, we can find its average rate of change (slope of secant line) and then use limits to find its instantaneous rate of change (slope of tangent line).

By the way, for any point on a function's graph, the tangent line is unique (there's only one) and hence the slope is unique. That is why (in the next section more than now) we will call this process a function. Do you remember what makes a relationship a function?

"Every x-value is associated with exactly one y-value."

We'll get there but first, some formal definitions and formulas.

vertical line test



(Any vertical line only goes through one point on the graph.)

Definition: Rate of Change and Slope of Tangent Line:

The **average rate of change in f** on the interval $[a, x]$ is the slope of the secant line between the points $(a, f(a))$ and $(x, f(x))$.

$$m_{\text{sec}} = \frac{f(x)-f(a)}{x-a}$$

Provided the limit exists.

The **instantaneous rate of change in f** at a is $m_{\tan} = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$. This is the slope of the tangent line at $x = a$.

$y - y_1 = m(x - x_1)$

★ The **equation of the tangent line** (as explored above) will always be $y - f(a) = m_{\tan}(x-a)$. ★

expl 2: For the following function, complete the following. $f(x) = -3x^2 - 5x + 1$

a.) Find the slope of the tangent line to f at $P = (1, -7)$.

b.) Find the equation of the tangent line. $a = 1$

c.) Plot f and the tangent line.

Take a to be 1. Find m_{tan}.

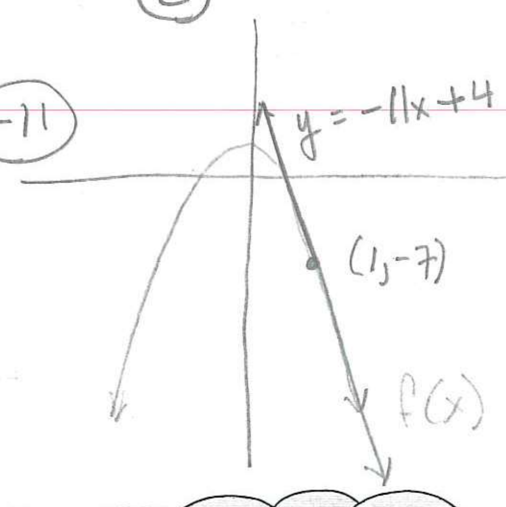
$f(1) = -3(1)^2 - 5(1) + 1$

$f(1) = -7$

(a)
$$m_{\tan} = \lim_{x\to1} \frac{f(x)-f(1)}{x-1}$$
$$= \lim_{x\to1} \frac{-3x^2-5x+1 - -7}{x-1}$$
$$= \lim_{x\to1} \frac{-3x^2-5x+8}{x-1}$$
$$= \lim_{x\to1} \frac{\cancel{(x-1)}(-3x-8)}{\cancel{(x-1)}}$$
$$= \lim_{x\to1}(-3x-8) = -3(1)-8 = -11$$

(b)
$$y - f(a) = m_{\tan}(x-a)$$
$$y - -7 = -11(x-1)$$
$$y + 7 = -11x + 11$$
$$y = -11x + 4$$

(c)



Use your calculator to graph on the window $[-5, 5] \times [-20, 10]$.

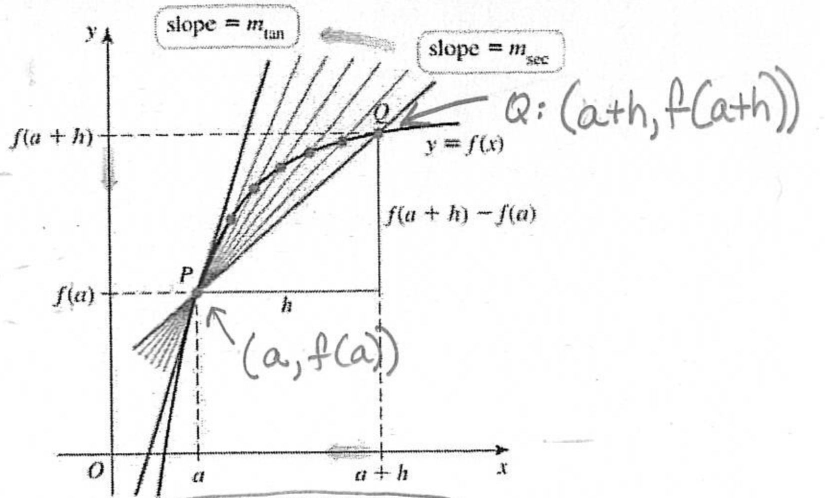
Alternative Definition Using the Difference Quotient:

An alternative formula we will use from time to time is presented here. Imagine P to be the point whose tangent line we are interested in. Then place a second point Q which is a little to its right.

We use h to denote "a little bit".

So, that we read P is the point where $x = a$ and Q is the point where $x = a$ plus a little bit, or rather $x = a + h$.

We find the slope between these two points which is the secant line from before.



$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ Difference Quotient

When we take the limit of this slope, as we let h (the space between P and Q) approach 0, we once again have the slope of the tangent line at P. We just got there a different way, didn't we?

Provided the limit exists.

As before, the slope of the secant line might be called the **average** rate of change of f and the limit (as h approaches 0) of this slope is the function's **instantaneous** rate of change.

expl 3a: Use this definition to find the slope of the tangent line at $P = (1, -1)$ for the function $f(x) = 3x^2 - 4x$.

~~You are given that $f(1) = -1$. You'll need to find $f(1+h)$.~~

$$M_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{3a^2 + 6ah + 3h^2 - 4a - 4h - (3a^2 - 4a)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3a^2} + 6ah + 3h^2 \cancel{- 4a} - 4h \cancel{- 3a^2} \cancel{+ 4a}}{h}$$

$$= \lim_{h \to 0} \frac{3h^2 + 6ah - 4h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3h + 6a - 4)}{\cancel{h}} = \lim_{h \to 0} (3h + 6a - 4)$$

$$= 3 \cdot 0 + 6a - 4$$

$$M_{tan} = 6a - 4$$

$f(x) = 3x^2 - 4x$

$f(a+h) = 3(a+h)^2 - 4(a+h)$

$= 3(a^2 + 2ah + h^2) - 4a - 4h$

$= 3a^2 + 6ah + 3h^2 - 4a - 4h$

For the point $(1, -1)$, $a = 1$, So $M_{tan} = 6 \cdot 1 - 4 = 2$

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expl 3b: Continue this example to find the equation of this tangent line at P. — $P: (1, -1)$

$$y - f(a) = m_{tan}(x-a)$$

$$y - -1 = 2(x-1)$$

$$y + 1 = 2x - 2$$

$$y = 2x - 3$$

Definition: Derivative of a Function at a Point:

More or less, to this point we have called this tangent line's slope just the function's "instantaneous rate of change". However, it is, in fact, the **derivative of f at a**, denoted $f'(a)$.
This assumes a is in the domain of f.

A "prime" has been added to the f.

$f'(a)$

You can use either formula we have seen here. So,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a)$ exists, then we say that f is **differentiable** at a.

expl 4: Evaluate the derivative of f at $a = 25$.
$f(s) = 2\sqrt{s} - 1$

Use the first formula taking s as the independent variable. Recall that a factor's conjugate can be useful.

find $f'(25)$

$$f'(25) = \lim_{s \to 25} \frac{f(s) - f(25)}{s - 25}$$

$$f(25) = 2\sqrt{25} - 1 = 9$$

$$(\sqrt{s} - 5)(\sqrt{s} + 5) \quad \text{FOIL}$$

$$= (\sqrt{s})^2 - \cancel{5\sqrt{s}} + \cancel{5\sqrt{s}} - 25$$

$$= s - 25$$

$$= \lim_{s \to 25} \frac{2\sqrt{s} - 1 - 9}{s - 25}$$

$$= \lim_{s \to 25} \frac{2\sqrt{s} - 10}{s - 25}$$

$$= \lim_{s \to 25} \frac{2(\sqrt{s} - 5)(\sqrt{s} + 5)}{(s - 25)(\sqrt{s} + 5)}$$

$$= \lim_{s \to 25} \frac{2\cancel{(s-25)}}{\cancel{(s-25)}(\sqrt{s} + 5)} = \lim_{s \to 25} \frac{2}{\sqrt{s} + 5} = \frac{2}{\sqrt{25} + 5} = \frac{2}{10} = \frac{1}{5}$$

direct substitution

$$K(10+h)^2 = k(10+h)(10+h)$$
$$= k(100+20h+h^2)$$
$$= 100k + 20kh + kh^2$$

expl 5: The gravitational attraction between two masses separated by a distance of x meters is inversely proportional to the square of the distance between them, or $\boxed{F(x) = k/x^2}$. Here, F is the force, measured in Newtons, and k is a real number. Find $F'(10)$, expressing your answer in terms of k. Interpret.

Use the second formula where $a = 10$.

(p95) $\rightarrow$ $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$F'(10) = \lim_{h \to 0} \frac{F(10+h) - F(10)}{h/1}$$

$$F(10+h) - F(10)$$
$$= \frac{k}{(10+h)^2} - \frac{k}{10^2}$$
$$= \frac{k}{(10+h)^2} - \frac{k}{100}$$
$$= \frac{100k - k(10+h)^2}{100(10+h)^2}$$
$$= \frac{100k - (100k + 20kh + kh^2)}{100(10+h)^2}$$
$$= \frac{\cancel{100k} - \cancel{100k} - 20kh - kh^2}{100(10+h)^2}$$
$$= \frac{h(-20k - kh)}{100(10+h)^2}$$

$$F'(10) = \lim_{h \to 0} \frac{\cancel{h}(-20k - kh)}{100(10+h)^2} \cdot \frac{1}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{-20k - kh}{100(10+h)^2}$$

(direct substitution)

$$= \frac{-20k - k(0)}{100(10+0)^2}$$

$$= \frac{-20k}{100(100)}$$

$$\boxed{F'(10) = -0.002k \ \text{N/m}}$$

When the masses are 10 m apart, the gravitational attraction between them is decreasing at a rate of 0.002 k N/m.

Your answer will employ k still as the constant particular to these masses. Include units and interpret.