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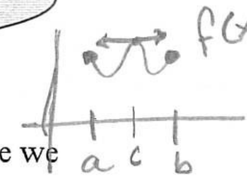
This will cement the relationship between average rate of change and instantaneous rate of change. It is used in the proofs of other theorems.

Calculus I
Class notes
The Mean Value Theorem (section 4.2)



Our main topic is, of course, the Mean Value Theorem (MVT). We will get to that once we explore another that is used to justify the MVT called Rolle's Theorem.

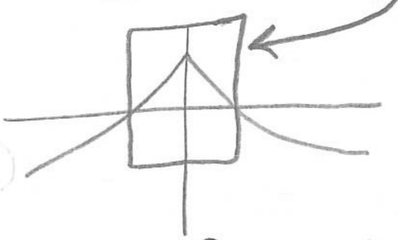
Theorem 4.3: Rolle's Theorem:
Let f be a continuous function on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. Then there is at least one point $x = c$ in (a, b) such that $f'(c) = 0$.

This is named after Michel Rolle, a French mathematician. He proved the case for polynomial functions in 1691.

expl 1: Determine if Rolle's Theorem applies to the following functions on the given intervals. If so, find the point(s) guaranteed to exist.

Graph on the window $[-3, 3] \times [-3, 3]$.

a.) $f(x) = 1 - x^{2/3}$, $[-1, 1]$ $\leftarrow$ $a = -1$ $b = 1$



Need to check
- f is cont on $[-1, 1]$ ✓
- f is diff on $(-1, 1)$ X
- $f(-1) = f(1)$

Recall, a function is *not* differentiable at a if
1) f is *not* continuous at a,
2) f has a corner at a, or
3) f has a vertical tangent at a.

(3, 2)

Since f is not diff at $x = 0$, then we cannot apply Rolles Thm.

b.) $g(x) = x^3 - x^2 - 5x - 3$, $[-1, 3]$

$g'(x) = 3x^2 - 2x - 5$

Need to check
- g is cont on $[-1, 3]$ ✓
- g is diff on $(-1, 3)$ ✓
- $g(-1) = g(3)$? ✓

$$g(-1) = (-1)^3 - (-1)^2 - 5(-1) - 3 = -1 - 1 + 5 - 3 = 0$$
$$g(3) = 3^3 - 3^2 - 5(3) - 3 = 27 - 9 - 15 - 3 = 0$$

$\rightarrow$ So, $g(-1) = g(3)$ ✓

So, Rolle's Thm applies

$$g'(x) = 0$$
$$3x^2 - 2x - 5 = 0$$
$$(3x - 5)(x + 1) = 0$$
$$3x - 5 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = 5/3 \qquad x = -1$$

Rolle's Thm guarantees the existence of $x = 5/3$ such that $g'(5/3) = 0$.

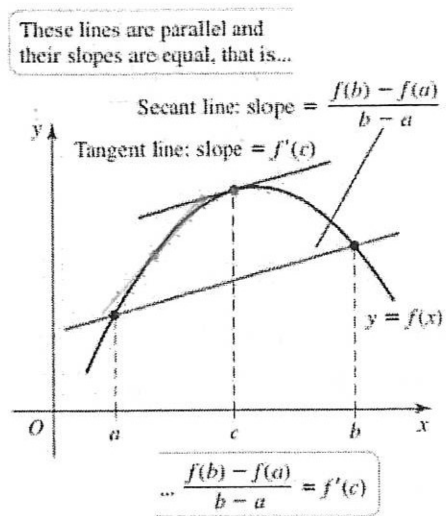
We only want values $x = c$ in the interval (a, b).

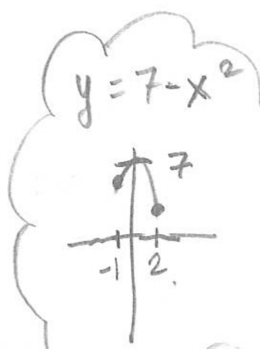
THEOREM 4.4 Mean Value Theorem
If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Notice the left-hand side of this equation is the *average* rate of change for the function in this interval. The MVT guarantees that there is some interior point such that its *instantaneous* rate of change is equal to it.

For the example function shown here, you can imagine the slopes of the tangent lines at any interior point. Most would *not* match the slope of the secant line from a to b. However, the MVT tells us that there *must* be *at least one* value in there whose slope does match. (It has been found and is shown at $x = c$.)





expl 2: Determine if the MVT applies to the function $f(x) = 7 - x^2$ on the interval $[-1, 2]$. If so, find the point(s) that are guaranteed to exist by the MVT.

Since f is cont on $[-1, 2]$ and diff on $(-1, 2)$, we see that MVT applies.

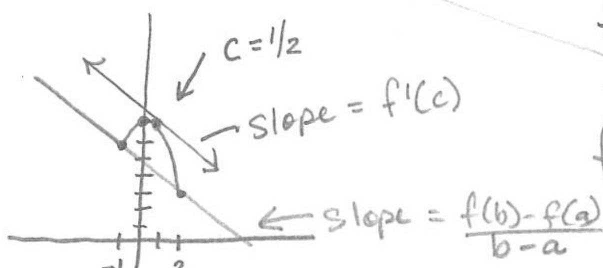
So, $a = -1, b = 2 \to f(-1) = 7 - (-1)^2 = 6$
$f(2) = 7 - 2^2 = 3$

To find c, solve $\frac{f(b) - f(a)}{b - a} = f'(c)$.

So $\frac{f(b) - f(a)}{b - a} = \frac{3 - 6}{2 - -1} = \frac{-3}{3} = -1$

$f(x) = 7 - x^2$
$f'(x) = -2x$
$f'(c) = -2c$

Know $-1 = -2c$
$c = 1/2$



Find the equation of the tangent line at your value of c and use your grapher to verify.

tangent line ($m = -1$, goes thru pt $(\frac{1}{2}, 6.75)$)
$y - 6.75 = -1(x - \frac{1}{2})$
$y - 6.75 = -x + \frac{1}{2}$
$y = -x + 7.25$

secant line ($m = -1$, pt $(2, 3)$)
$y - 3 = -1(x - 2)$
$y - 3 = -x + 2$
$y = -x + 5$

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expl 3: Avalanche forecasters measure the temperature gradient $\frac{dT}{dh}$, the rate at which the temperature in a snowpack T changes with respect to its depth h. A large temperature gradient may lead to a weak layer in the snowpack, possibly causing an avalanche. If this temperature gradient exceeds 10 °C/m anywhere in the snowpack, a weak layer is indicated and the risk of an avalanche increases. Assume the temperature function is continuous and differentiable.

a.) An avalanche researcher takes two temperature measurements. At the surface ($h = 0$), the temperature is –16 °C. At a depth of 1.1 meters, the temperature is –2 °C. Using the MVT, what can be said about the temperature gradient in the interval (0, 1.1) and the risk of an avalanche?

$(h, T(h))$

$(0, -16)$

$(1.1, -2)$

$a = 0$

$b = 1.1$

MVT: $\frac{T(b)-T(a)}{b-a} = T'(c)$ for some $c \in (0, 1.1)$

$\frac{T(b)-T(a)}{b-a} = \frac{-2 - -16}{1.1 - 0} \approx 12.7$ °C/m

Somewhere between $h=0$ and $h=1.1$, this gradient will be 12.7 °C/m - which is more than 10 °C/m so there is a risk of avalanche.

The function $T(h)$ is the temperature at depth h.

b.) At another location, temperature readings are taken. At the surface ($h = 0$), the temperature is –12 °C. At a depth of 1.4 meters, the temperature is –1 °C. Using the MVT, what can be said about the temperature gradient in the interval (0, 1.4) and the risk of an avalanche?

$(0, -12)$

$(1.4, -1)$

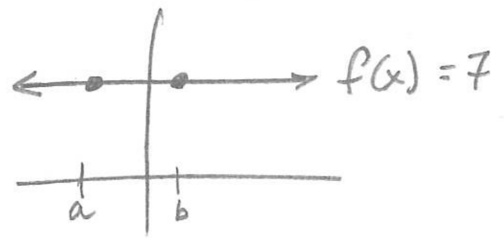
MVT: $\frac{T(b)-T(a)}{b-a} = \frac{-1 - -12}{1.4 - 0} \approx 7.9$ °C/m

Since this is less than 10 °C/m, we see no evidence of avalanche risk.

A result that is *not* more than 10 °C/m does *not* necessarily mean there is *no* risk. We simply do *not* have evidence for it.

Consequences of the Mean Value Theorem:
These observations will help as we study antiderivatives later in the chapter.

First, let's look at a constant function. Draw any you like to the right. Label it in function notation.



Find the value of $\dfrac{f(b)-f(a)}{b-a}$ for any two values a and b. Do you see why you would always get the same value?

slope $= \dfrac{f(b)-f(a)}{b-a} = 0$

Sidebar: Logic: The Converse:
A conditional statement can be thought of as, "If p, then q." Its **converse** is "If q, then p."

If we know a statement is true, we *cannot* say if the statement's converse is also true. It may be true *but* it may *not* be true. (Earlier, we saw how a statement's contrapositive is always true if the statement is true. Here, we dabble a bit more in logic.)

We can see this by considering an example. Assume the statement "If it rains, then I'll drive you home." is true. Is the converse "If I drive you home, then it rained." also true?

The answer is *no*; I could be driving you home for another reason.

Do you recall learning that $\dfrac{d}{dx}(c)=0$?

Back to Calculus:
So, as seen at the top of this page, we know that if a function is a constant function, that is, $f(x)=C$ for some real number C, then $f'(x)=0$ for all x. This statement's converse also happens to be true. The book lays this bad boy out thusly…

> **THEOREM 4.5 Zero Derivative Implies Constant Function**
> If f is differentiable and $f'(x) = 0$ at all points of an open interval I, then f is a constant function on I.

"If and Only If" Language:
We could state this as "A function f is differentiable and $f'(x)=0$ for all points of an open interval **if and only if** f is a constant function on that interval." One condition implies the other.

The statement "p if and only if q" is the same as "If p, then q." and, at the same time, "If q, then p." This is called a **biconditional** statement.

Contrapositive: If not q, then not p.
(Contrapositives are always true when the orig statement is true.)

contrapositive: If f and g don't differ by a constant, then $f' \neq g'$.

converse: If f and g differ by a constant, then $f' = g'$.

The MVT helps prove Theorem 4.5. A direct consequence of Theorem 4.5 is below. Its converse and contrapositive (used in the next example) are also true. Proofs are given in the book.

THEOREM 4.6 Functions with Equal Derivatives Differ by a Constant
If two functions have the property that $f'(x) = g'(x)$, for all x of an open interval I, then $f(x) - g(x) = C$ on I, where C is a constant; that is, f and g differ by a constant.

orig. statement: "If $f' = g'$, then f and g differ by a constant."

This one is a good one and helps enormously when we work with antiderivatives which, as the name sort of implies, helps us *reverse* what the derivative does. For now, we will work this example.

expl 4: *Without* evaluating derivatives, determine which of the functions listed to the right have the same derivative as $f(x) = x^{10}$.

$g(x) = 2x^{10}$

$h(x) = x^{10} + 5$

$p(x) = x^{10} + \ln 2$

$k(x) = x^{10} + 5x$

→ Find $f - g = x^{10} - 2x^{10} = -x^{10}$
This is not a constant (because it has "x" in it). So according to the contrapositive of Thm 4.6, we know $f' \neq g'$.

"Differing by a constant" means that if you subtract them, you would get a constant.

→ Find $f - h = x^{10} - (x^{10} + 5) = -5$ which is a constant. So, according to the converse of Thm 4.6, $f' = h'$.

→ Find $f - p = x^{10} - (x^{10} + \ln 2) = -\ln 2$ which is a constant. So, according to the converse of Thm 4.6, $f' = p'$.

→ Find $f - k = x^{10} - (x^{10} + 5x) = -5x$ which is not a constant. Hence, according to the contrapositive of Thm 4.6, we know $f' \neq k'$.

You could check yourself by using the derivative rules.