2:00

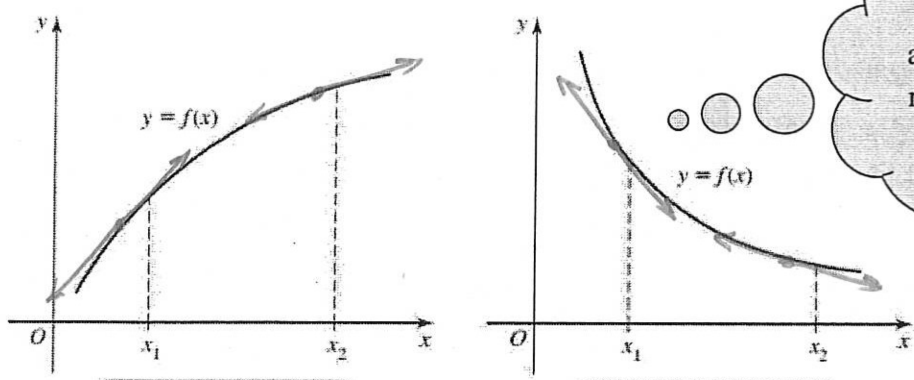
The first and second derivatives of a function can tell you a lot about its graph.

Calculus I
Class notes
The First and Second Derivative Tests (section 4.3)

The concepts of increasing and decreasing as well as concavity come back from algebra here. We will understand how the first and second derivatives help us determine these important characteristics of a function's graph. First, let's review.

> DEFINITION **Increasing and Decreasing Functions**
> Suppose a function f is defined on an interval I. We say that f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say that f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

An increasing function will rise as you travel left to right. A decreasing function will fall.



f increasing: $f(x_2) > f(x_1)$ whenever $x_2 > x_1$
(a)

f decreasing: $f(x_2) < f(x_1)$ whenever $x_2 > x_1$
(b)

As you look at these graphs, imagine some tangent lines. Go crazy and draw a few in. What do you notice about those of an increasing graph versus those tangent lines on the decreasing graph?

Increasing fncs will have positive-sloped tangent lines.

Decreasing fncs will have negatively-sloped tangent lines.

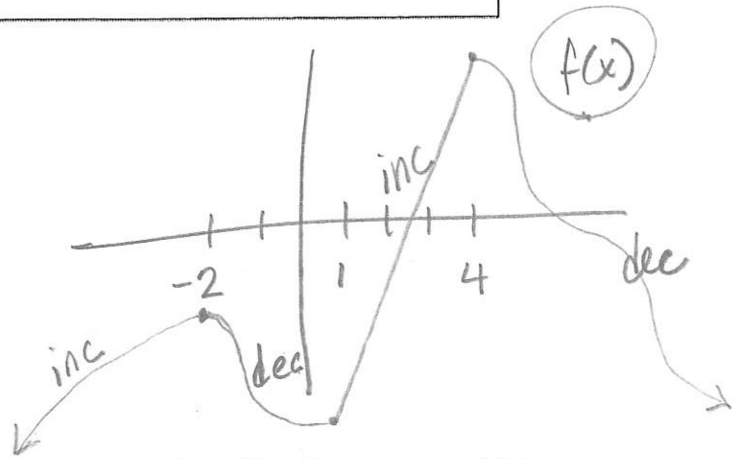
This realization brings us to our first theorem of this section. Once again, the proof is given in the book but your examples should be pretty convincing.

And, $f'(x)$ is the slope of this tangent line.

THEOREM 4.7 Test for Intervals of Increase and Decrease
Suppose f is continuous on an interval I and differentiable at all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

expl 1: Sketch a graph of a function $f(x)$ such that $f'(x)<0$ on $(-2, 1)$ and $(4, \infty)$ but $f'(x)>0$ on $(-\infty, -2)$ and $(1, 4)$.

Start with an xy-plane and mark off x-values.



On homework, you will see these questions as multiple-choice. What if you were told that $f'(-2) = 0$? What would that tell you about the function's graph?
slope of tang line = 0 (maybe a max or min)

expl 2: Find intervals on which f is increasing or decreasing.

Dust off your factoring skills.

$f(x) = x^4 - 4x^3 + 4x^2$

We notice f is cont and diff because fnc is polynomial.

Find $f'(x) = 4x^3 - 12x^2 + 8x$

$0 = 4x^3 - 12x^2 + 8x$

$0 = 4x(x^2 - 3x + 2)$

$0 = 4x(x-2)(x-1)$

$4x = 0$ or $x - 2 = 0$ or $x - 1 = 0$

$x = 0$ $\quad$ $x = 2$ $\quad$ $x = 1$

Critical points

$x = -2$ $\quad$ $x = 1/2$ $\quad$ $x = 3/2$ $\quad$ $x = 4$

$f'(x) < 0$ $\quad 0 \quad$ $f'(x) > 0$ $\quad 1 \quad$ $f'(x) < 0$ $\quad 2 \quad$ $f'(x) > 0$

We'll make a "sign graph".

$f'(x) = 4x(x-2)(x-1)$

$f'(-2) = \text{neg} \cdot \text{neg} \; \text{neg} \rightarrow \text{neg}$

$f'(1/2) = \text{pos} \cdot \text{neg} \cdot \text{neg} \rightarrow \text{pos}$

$f'(3/2) = \text{pos} \cdot \text{neg} \cdot \text{pos} \rightarrow \text{neg}$

$f'(4) = \text{pos} \cdot \text{pos} \cdot \text{pos} \rightarrow \text{pos}$

So, $f(x)$ decreases on $(-\infty, 0)$ and $(1, 2)$.
And $f(x)$ increases on $(0, 1)$ and $(4, \infty)$.

Sign Graphs:

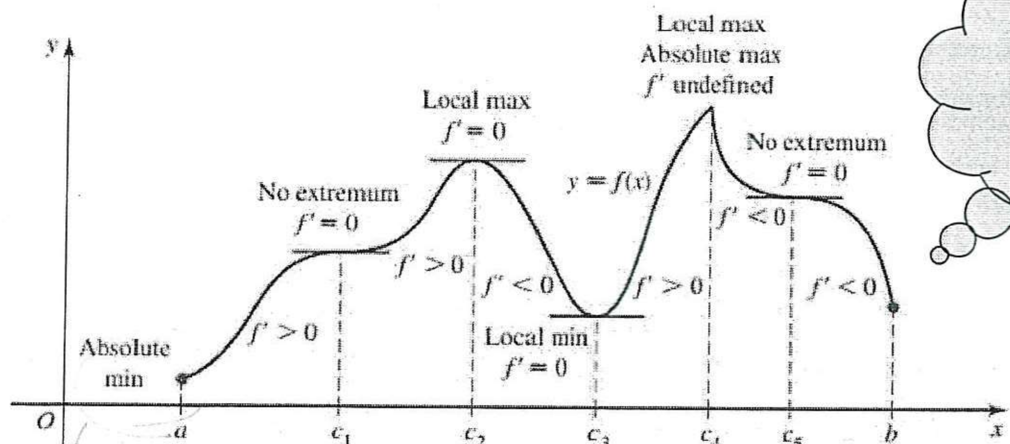
The process, as seen in the previous example, will be to find critical points (where $f'(x)=0$ or is undefined) and then to fill out a **sign graph** using these values as endpoints of intervals. For each interval, we need only find out if $f'(x)$ is positive or negative. Hence, we only need one value in each interval to test. If one $f'(x)$ is positive in the interval, then *all* $f'(x)$ values will be.

You can always graph on the calculator to check your conclusions.

$f'(x)$

Local Minimums and Maximums:

A critical point *may* be where a local min or max occurs but it does *not* have to be. Check out this graph where we have both $f'(c)=0$ and $f'(c)$ is undefined.



Find the points where $f'(c)=0$ but there is *not* a max *nor* min.

$f(a)$ is not a local min because it's an endpoint (not an interior point)

Find the local maxes and mins above. Do you notice something interesting happening on either side of those extrema? We have the **First Derivative Test** to put it in words.

THEOREM 4.8 First Derivative Test

Assume f is continuous on an interval that contains a critical point c, and assume f is differentiable on an interval containing c, except perhaps at c itself.

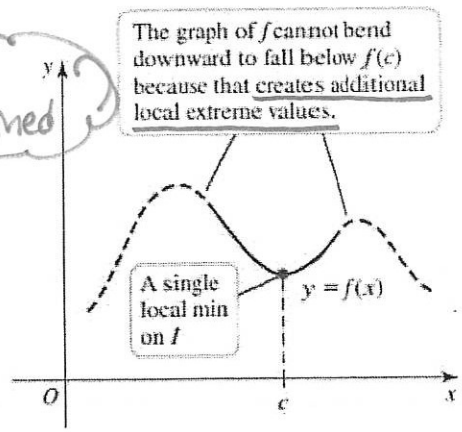
- If f' changes sign from positive to negative as x increases through c, then f has a **local maximum** at c. (x-values C_2 and C_4)
- If f' changes sign from negative to positive as x increases through c, then f has a **local minimum** at c. (x-value C_3)
- If f' is positive on both sides near c or negative on both sides near c, then f has no local extreme value at c. (x-values C_1 and C_5)

Connection Between Local and Absolute Extrema:
This one is useful when interpreting a function's graph.

> **THEOREM 4.9 One Local Extremum Implies Absolute Extremum**
> Suppose f is continuous on an interval I that contains exactly one local extremum at c.
> - If a local maximum occurs at c, then $f(c)$ is the absolute maximum of f on I.
> - If a local minimum occurs at c, then $f(c)$ is the absolute minimum of f on I.

Here is a good example of why this theorem must be true. Would you agree with it?

Crit Pts.
$f'(x)=0$ or undefined



expl 3: For the following function, locate the critical points of f in the interval $[-2, 4]$. Use the First Derivative Test to find local extrema. Lastly, check if Theorem 4.9 applies and, if so, use it to identify an absolute extremum.

$f(x) = 2x^3 + 3x^2 - 12x + 1$

We'll make a "sign graph".

Critical points:

$f'(x) = 6x^2 + 6x - 12$

$0 = 6x^2 + 6x - 12$

$0 = 6(x^2 + x - 2)$

$0 = 6(x+2)(x-1)$

Crit Pts: $x+2=0$ or $x-1=0$

$x=-2$ $\quad$ $x=1$

Critical Points

interval $[-2,4]$

$x=-4$ $\quad$ $x=0$ $\quad$ $x=3$

$f'(x)>0$ $\quad$ -2 $\quad$ $f'(x)<0$ $\quad$ 1 $\quad$ $f'(x)>0$

First Der. Test

Test says $f(-2)$ is local max.
But $x=-2$ is an endpoint of interval $[-2,4]$ → so not considered a local max.
Test says $f(1)$ is local min.

$f'(x) = 6(x+2)(x-1)$

$f'(-4)$ = pos · neg · neg → pos

$f'(0)$ = pos pos neg → neg

$f'(3)$ = pos pos pos → pos

Do you recall why we say a local max does *not* occur at $x=-2$?

More room on next page.

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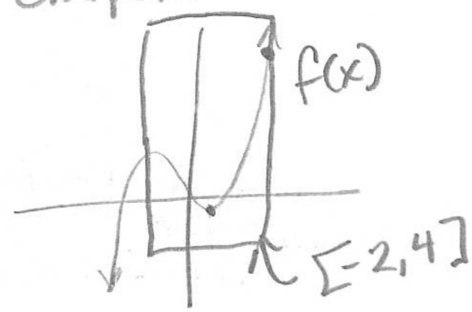
When you test a value in an interval, remember that you only need to know if it will be positive or negative. You do *not* need to know its value.

expl 3 continued: Theorem 4.9 will give us one absolute extremum. However, you will need some more work to find the other. Recall the procedure for finding absolute extrema on a closed interval from earlier. (4.1)

Since $f(1)$ is the only local min on $[-2,4]$, Thm 4.9 tells us it's also an absolute minimum.

From 4.1, check crit pt ($x=1$) and endpoints to be abs. extrema

-2	1	4
$f(-2)=21$	$f(1)=-6$	$f(4)=129$
	Abs min	Abs max



Graph on the window [-10, 10] x [-10, 150] to check. Remember we are interested in the interval [-2, 4]. ✓

expl 4: Verify that this function satisfies the conditions of Theorem 4.9 on its domain. Then find the location and value of the absolute extremum guaranteed by the theorem.

$h(x) = 4x^2 - 3x + 4$

What is the domain? How do you know f is continuous? Do you have a rough graph in your head?

Since $h(x)$ is a polynomial fnc, we know it is continuous on $(-\infty, \infty)$. Since $h(x)$ is quadratic and is a parabola, we know it has exactly one local extremum.

Find local extremum

$h'(x) = 8x - 3$

$0 = 8x - 3$

$x = 3/8$

$x=0$ | $\frac{3}{8}$ | $x=1$

$h'(x) < 0$ | $h'(x) > 0$

$h'(x) = 8x - 3$

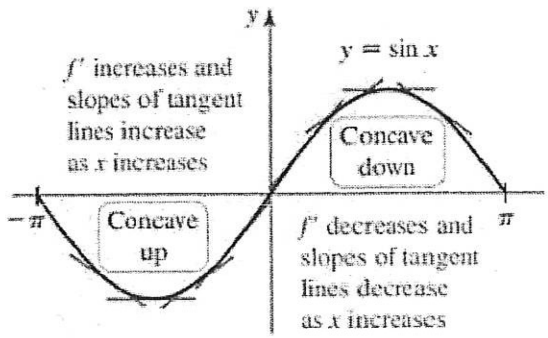
$h'(0) = 8 \cdot 0 - 3 = -3$ neg

$h'(1) = 8 \cdot 1 - 3 = 5$ pos

From the 1st Der Test, we see that $h(3/8)$ is a local minimum. Since f is cont and has exactly one local extrema, we know (Thm 4.9) that $h(3/8)$ is abs min.

Concavity and Inflection Points:

Take a look at this graph. Algebra taught us the concepts of concave up and down. Now, we apply calculus to this graph.



There is a point in the middle where the graph changes from concave up to concave down. This point is an **inflection point**. We will also see that the slopes of these tangent lines stop increasing and start decreasing.

Below, we see how the derivative plays a role.

> DEFINITION **Concavity and Inflection Point**
>
> Let f be differentiable on an open interval I. If f' is increasing on I, then f is **concave up** on I. If f' is decreasing on I, then f is **concave down** on I.
>
> If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an **inflection point** at c.

Now, this mentions that "If f' is increasing [or decreasing] on I …"

How does calculus tell us when a function is increasing or decreasing? That, you say, involves finding the function's derivative. What is the derivative of f'? And so, we have this test for concavity.

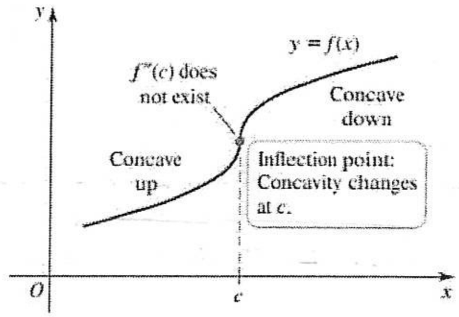
> **THEOREM 4.10 Test for Concavity**
>
> Suppose f'' exists on an open interval I.
>
> - If $f'' > 0$ on I, then f is concave up on I.
> - If $f'' < 0$ on I, then f is concave down on I.
> - If c is a point of I at which f'' changes sign at c (from positive to negative, or vice versa), then f has an inflection point at c.

Do *not* read too much into this. It is *not necessarily true* that if $f''(c)=0$, then f'' *must* have changed sign and you would find an inflection point. Can you think of an example where $f''(c)=0$ for some c but f has *no* inflection point there?

?

When the Second Derivative Does *Not* Exist:

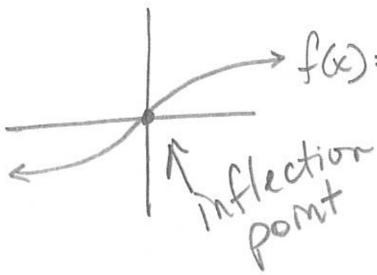
An inflection point may also occur when the second derivative does *not* exist. Here is the graph of such a function.



(c)

Another example is the function $f(x) = \sqrt[3]{x} = x^{1/3}$. Do you have its graph in your head?

Find the second derivative of $f(x) = \sqrt[3]{x} = x^{1/3}$ to show it does *not* exist at $x = 0$ (its inflection point).



$f(x) = x^{1/3}$

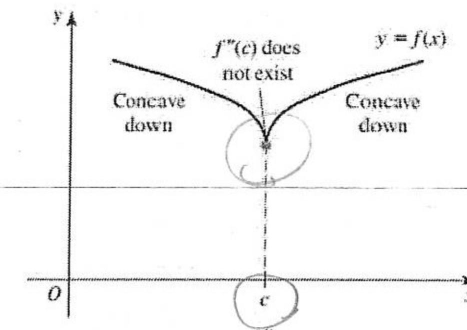
$f'(x) = \frac{1}{3}x^{-2/3}$ (Power Rule)

$f''(x) = \frac{1}{3} \cdot \frac{-2}{3} \cdot x^{-5/3}$ (Power Rule)

$f''(x) = -\frac{2}{9}x^{-5/3}$ or $= \frac{-2}{9x^{5/3}}$

Notice $f''(0)$ dne.

Here's another example of a function whose second derivative does *not* exist.



(d)

Here, we do *not* call the point in the middle an inflection point. Do you know why?

The fnc does not change concavity at x=c.

2:00

$(x+1)^{5/2} = \sqrt{(x+1)^5}$ is only defined for $x \in [-1, \infty)$

expl 5: Determine intervals on which the following function is concave up and concave down. Identify inflection point(s). Consider the domain interval of $[-1, \infty)$.

$g(x) = 8(x+1)^{5/2}(4x-9)$

We need the Product Rule here. Some tricky algebra ahead. Be you careful!

$\frac{d}{dx}(fg) = f'g + fg'$

$$g'(x) = 8 \cdot \tfrac{5}{2}(x+1)^{3/2}(4x-9) + 8(x+1)^{5/2} \cdot 4$$

$$= 20(x+1)^{3/2}(4x-9) + 32(x+1)^{5/2}$$

$$= (x+1)^{3/2}\left[20(4x-9) + 32(x+1)^{2/2}\right]$$

$$g'(x) = (x+1)^{3/2}(80x - 180 + 32x + 32)$$

$$g'(x) = (x+1)^{3/2}(112x - 148)$$

Simplify the first derivative as much as you can before finding the second.

$$g''(x) = \tfrac{3}{2}(x+1)^{1/2} \cdot 1 (112x-148) + (x+1)^{3/2}(112)$$

$$g''(x) = (x+1)^{1/2}(168x - 222) + 112(x+1)^{3/2}$$

$$= (x+1)^{1/2}(168x - 222 + 112(x+1))$$

$$= (x+1)^{1/2}(168x - 222 + 112x + 112)$$

$$g''(x) = (x+1)^{1/2}(280x - 110)$$

$$0 = (x+1)^{1/2}(280x - 110)$$

$(x+1)^{1/2} = 0$ or $280x - 110 = 0$

$\sqrt{x+1} = 0$ $\qquad x = 110/280$

$x + 1 = 0$ $\qquad x = 11/28$

$x = -1$

Consult Theorem 4.10 and use a sign graph.

interval $[-1, \infty)$

interval is not considered

$x = 0$ $\qquad x = 1$

-1 $\qquad g''(x) < 0$ $\qquad \frac{11}{28}$ $\qquad g''(x) > 0$

$g''(0) = (0+1)^{1/2}(280 \cdot 0 - 110) = \text{pos} \cdot \text{neg} \to \text{neg}$

$g''(1) = (1+1)^{1/2}(280 \cdot 1 - 110) \to \text{pos} \cdot \text{pos} \to \text{pos}$

So, Thm 4.10 says g is concave down on $(-1, 11/28)$ and concave up on $(\frac{11}{28}, \infty)$. And there's an inflection point at $c = 11/28$.

The homework will require exact answers. Graph on [-5, 5] x [-250, 100] or similar to check yourself.

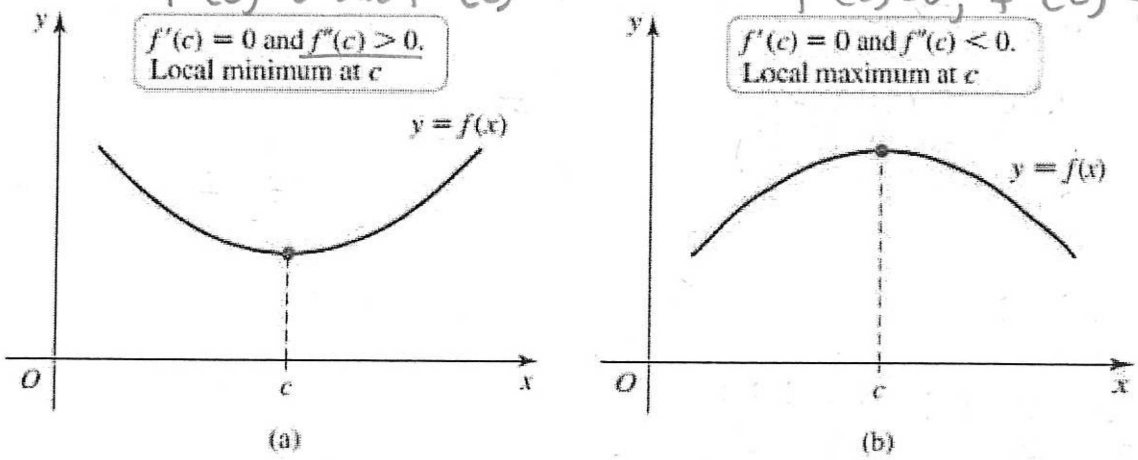
graph on pg 9

want graph still

Second Derivative Test:

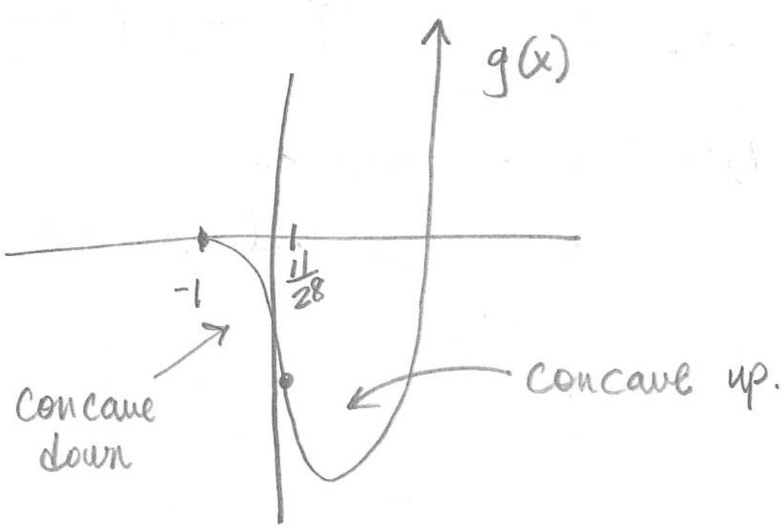
You may remember from algebra the connection between concavity and local maxes and mins. Take a look at these pictures, noticing the left graph is concave up and the right is concave down.

$f'(c)=0$ and $f''(c)>0$ $\qquad\qquad$ $f'(c)=0$, $f''(c)<0$



The Second Derivative Test (proven, in part, with the First Derivative Test) is our last piece of this puzzle.

> **THEOREM 4.11 Second Derivative Test for Local Extrema**
> Suppose f'' is continuous on an open interval containing c with $f'(c) = 0$.
> - If $f''(c) > 0$, then f has a local minimum at c (Figure 4.38a).
> - If $f''(c) < 0$, then f has a local maximum at c (Figure 4.38b).
> - If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, a local minimum, or neither at c.



find $f'(c) = 0$

expl 6: Find the critical points for the following function. Then use the Second Derivative Test to locate local maxes and mins.

$p(t) = 2t^3 + 3t^2 - 36t$

Differentiate and factor.

$p'(t) = 6t^2 + 6t - 36$

$0 = 6t^2 + 6t - 36$

$0 = 6(t^2 + t - 6)$

$0 = 6(t+3)(t-2)$

$t+3=0$ or $t-2=0$

$t=-3$ $\quad$ $t=2$

Critical points (c from Thm)

$c = -3, 2$

Do *not* gloss over the conditions of the test.

Find $p''(t) = 12t + 6$

We know $p''(t)$ is continuous over $(-\infty, \infty)$ because it's a polynomial.

Remember we do *not* need values for p'', only if it's positive or negative.

Find $p''(-3) = 12(-3) + 6 < 0$

$p''(2) = 12 \cdot 2 + 6 > 0$

So, by the 2nd Der. Test we see

$p(-3)$ is a local maximum

and $p(2)$ is a local minimum.

Recap of Derivative Properties

This section has demonstrated that the first and second derivatives of a function provide valuable information about its graph. The relationships among a function's derivatives and its extreme values and concavity are summarized in Figure 4.41.

Thm 4.7

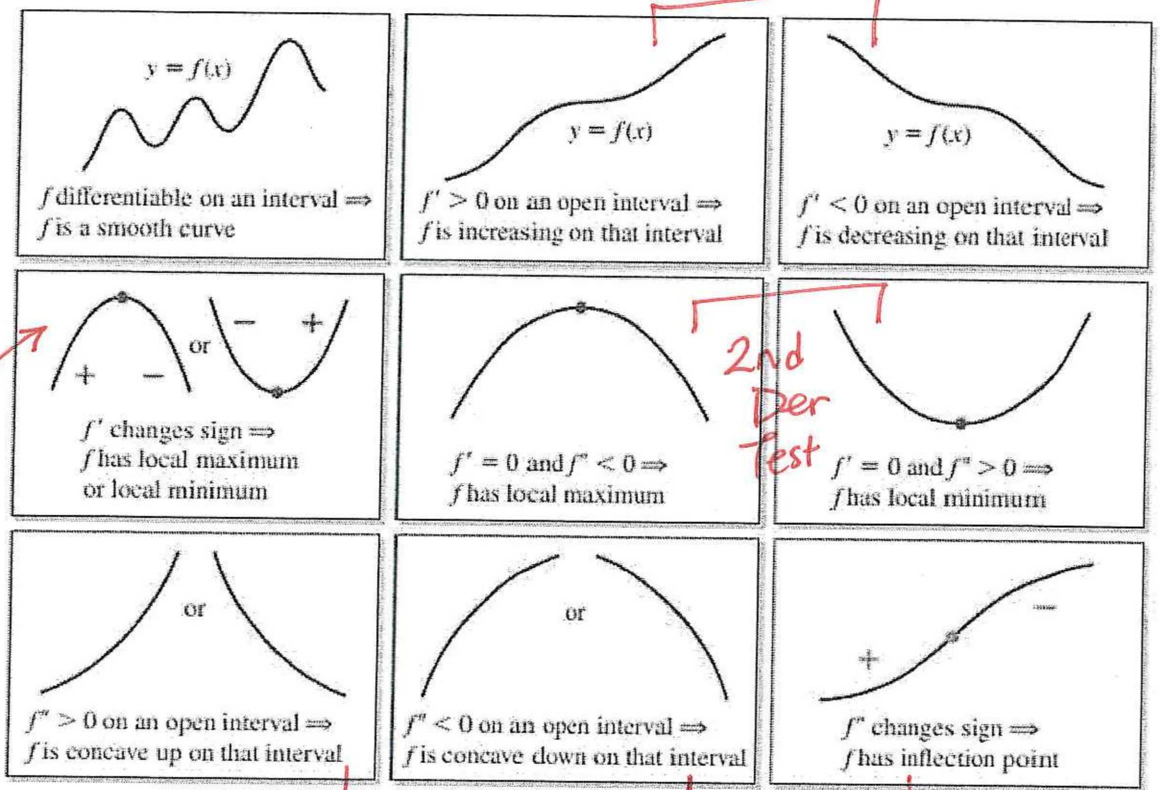


Figure 4.41

Thm 4.10