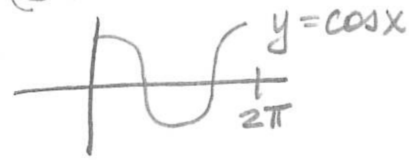
lo-pee-tal

2:00

We attack limit problems on which we have previously hit a dead end.

Calculus I
Class notes
L'Hôpital's Rule (section 4.7)

Recall, we could *not* find $\lim_{x\to 0}\frac{\sin x}{x}$ directly because direct substitution gets us $\frac{0}{0}$ and that does us little good. This $\frac{0}{0}$ is called an **indeterminate** form. But, look at this!



Take the derivatives of top and bottom and we get $\dfrac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} = \dfrac{\cos x}{1}$. Now, take the limit (as x approaches 0) of this and what do we get?

$\lim_{x\to 0}\frac{\cos x}{1} = \frac{1}{1} = 1$

Recall, we were told in a previous section that this limit was, in fact, 1. Now we know why. It is an example of an ingenious method called L'Hôpital's Rule. We will see various versions of it but they pretty much all say the same thing.

With it, we shall conquer limits involving the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, and $\infty-\infty$.

Named after a French mathematician Guillaume de l'Hôpital but first presented by Swiss mathy Johann Bernoulli.

THEOREM 4.12 L'Hôpital's Rule
Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is replaced with $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

This is super cool. Take the derivatives of the top and bottom and try direct substitution again. If it doesn't work, you can keep going…

The proof is rather clever and relatively easy to follow. Look it up in the book!

First, verify you have an indeterminate form.

expl 1: Find the following limit.

$$\lim_{x\to -1}\frac{x^4+x^3+2x+2}{x+1}$$

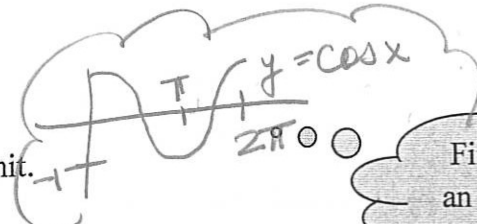
Verify: $\lim\limits_{x\to -1}\dfrac{x^4+x^3+2x+2}{x+1}$

direct sub $= \dfrac{(-1)^4+(-1)^3+2(-1)+2}{-1+1} = \dfrac{0}{0}$ ✓

$$= \lim_{x\to -1}\frac{4x^3+3x^2+2}{1}$$

direct sub $= \dfrac{4(-1)^3+3(-1)^2+2}{1} = \dfrac{-4+3+2}{1} = \dfrac{1}{1} = 1$

Graph to check!



expl 2: Find the following limit.

First, verify you have an indeterminate form.

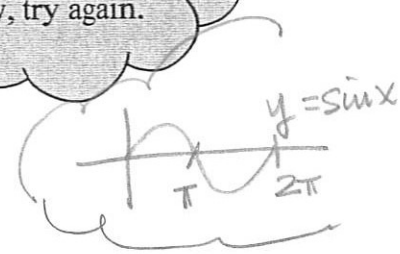
$$\lim_{x\to \pi}\frac{\cos x+1}{(x-\pi)^2}$$

Verify: $\lim\limits_{x\to \pi}\dfrac{\cos x+1}{(x-\pi)^2} =$ direct sub $= \dfrac{\cos \pi+1}{(\pi-\pi)^2} = \dfrac{0}{0}$ ✓

$$\lim_{x\to \pi}\frac{\cos x+1}{(x-\pi)^2}$$

If at first you don't succeed, try, try again.

$= \lim\limits_{x\to \pi}\dfrac{-\sin x}{2(x-\pi)^1\cdot 1} =$ direct sub $= \dfrac{-\sin \pi}{2(\pi-\pi)} = \dfrac{0}{0}$



L Hop Again

$= \lim\limits_{x\to \pi}\dfrac{-\cos x}{2} =$ direct sub $\dfrac{-\cos \pi}{2} = \dfrac{1}{2}$

2:00

$\frac{d}{dx}(e^x) = e^x$

Chain Rule: Paul Dawkins handout from 3.3

$\frac{d}{dx}\left(e^{g(x)}\right) = g'(x) \cdot e^{g(x)}$

The Indeterminate Form ∞/∞:

> **THEOREM 4.13 L'Hôpital's Rule (∞/∞)**
> Suppose f and g are differentiable on an open interval I containing a, with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then
> $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$
> provided the limit on the right exists (or is $\pm\infty$). The rule also applies for $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

First, verify you have an indeterminate form.

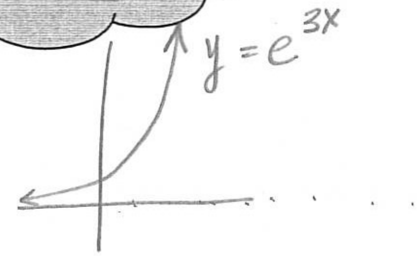
Can you picture the graph of $y = e^{3x}$?

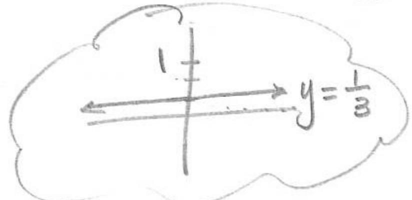
expl 3: Find the following limit.

$$\lim_{x\to\infty} \frac{e^{3x}}{3e^{3x}+5} = \frac{\lim_{x\to\infty} e^{3x}}{\lim_{x\to\infty} (3e^{3x}+5)} = \frac{\infty}{\infty}$$

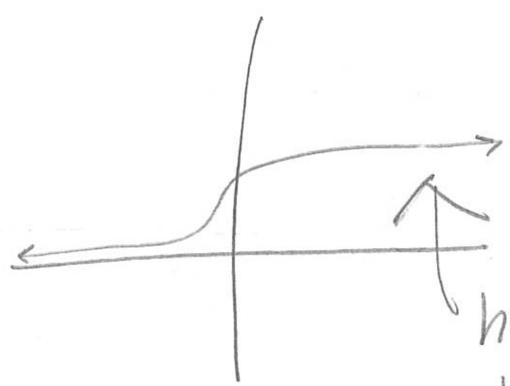
L'Hop

$$= \lim_{x\to\infty} \frac{3\cancel{e^{3x}}}{9\cancel{e^{3x}}} = \lim_{x\to\infty} \left(\frac{1}{3}\right) = \frac{1}{3}$$





Check by graphing $y = \dfrac{e^{3x}}{3e^{3x}+5}$.
A good window is [–5, 50] x [–1, 1].
Trace to the right end and what do you see?



hor. asymp at $y = \frac{1}{3}$.

$\csc x = \dfrac{1}{\sin x}$

The Indeterminate Form $0 \cdot \infty$:

First, verify you have an indeterminate form. This will be $0 \cdot \infty$. Or is it?

Rewrite the function whose limit we want.

expl 4: Find the following limit.

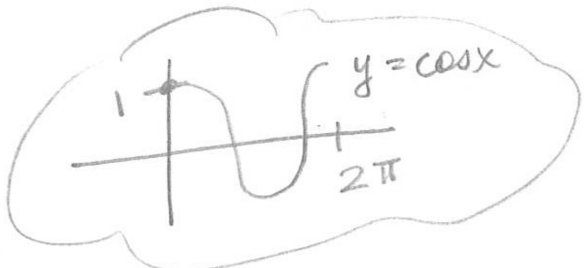
$\lim\limits_{x \to 0} (x \cdot \csc x)$

$= \lim\limits_{x \to 0} \left(\dfrac{x}{1} \cdot \dfrac{1}{\sin x} \right) = \lim\limits_{x \to 0} x \cdot \lim\limits_{x \to 0} \dfrac{1}{\sin x} = 0 \cdot \infty$

We'll rewrite this so it appears to in the form $0/0$.

$= \lim\limits_{x \to 0} \dfrac{x}{\sin x}$ direct sub $= \dfrac{0}{\sin 0} = \dfrac{0}{0}$ ✓

L'Hop $\rightarrow \lim\limits_{x \to 0} \dfrac{1}{\cos x} =$ direct sub $\rightarrow \dfrac{1}{\cos 0} = \dfrac{1}{1} = 1$



We will see these in the form of $\lim\limits_{x \to a} (f(x) - g(x))$.

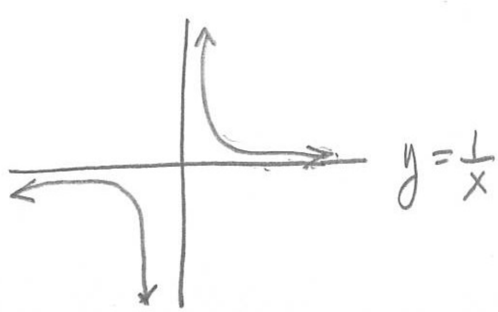
The Indeterminate Form $\infty - \infty$:

These can be tricky and deceptive. The book suggests we convert a limit as $x \to \infty$ to a limit as $t \to 0^+$ or vice versa. This is a trick that involves, as you see, a change of variables.

This works because if $x \to \infty$, then $1/x \to 0^+$. Do you agree with that? Do you see why we say $1/x$ approaches 0 from the *right*? ✓

Essentially, we are letting $t = 1/x$ and finding the limit as $t \to 0^+$ instead of finding the original limit as $x \to \infty$.

Draw a quick graph of $y = 1/x$ to help explore this.



$\sqrt{a^4+b^2} \not\overset{?}{=} a^2+b$

Identify $f(x)$ and $g(x)$ in this difference. Break this limit apart and see it as $\infty - \infty$. ✓

Put it back together and play around with it until it looks like 0/0.

~~If we let $t = 1/x$, then what's x^2, x^4, or $16x^2$?~~

expl 5: Find the following limit.

$$\lim_{x\to\infty}\left(x^2 - \sqrt{x^4+16x^2}\right)$$

We will give the book's suggestion a try. Using a change of variables to rewrite this limit is good practice as this is a useful trick. (You *could* do this problem as is without the book's suggestion. Either way, we need a fair bit of algebra.)

Verify: $\lim_{x\to\infty} x^2 - \lim_{x\to\infty}\sqrt{x^4+16x^2} = \infty - \infty$

$ab - ac = a(b-c)$

$$\lim_{x\to\infty}\left(x^2 - \sqrt{x^4+16x^2}\right)$$

$$= \lim_{x\to\infty}\left(x^2 - \sqrt{x^4}\sqrt{1+16/x^2}\right)$$

$$= \lim_{x\to\infty}\left(x^2 - x^2\sqrt{1+16/x^2}\right)$$

$$= \lim_{x\to\infty}\left(x^2\left(1-\sqrt{1+16/x^2}\right)\right)$$

$$= \lim_{x\to\infty} x^2 \cdot \lim_{x\to\infty}\left(1-\sqrt{1+\underset{\to 0}{16/x^2}}\right)$$

$= \infty \cdot 0$ This is indeterminate form from #4 so, we need to turn it into form 0/0.

$$= \lim_{x\to\infty}\frac{1-\sqrt{1+16/x^2}}{1/x^2}$$ This has the indeterminate form 0/0. ✓

Let $t = 1/x$, then $\frac{16}{x^2} = \frac{16}{1}\left(\frac{1}{x}\right)^2 = 16t^2$ and $\frac{1}{x^2} = \left(\frac{1}{x}\right)^2 = t^2$

and as $x\to\infty$, $t\to 0^+$.

$$= \lim_{t\to 0^+}\frac{1-\sqrt{1+16t^2}}{t^2}$$ (This is still 0/0. So, use l'Hopital's Rule)

$$= \lim_{t\to 0^+}\frac{1-(1+16t^2)^{1/2}}{t^2} \xrightarrow{\text{L'Hop}} \lim_{t\to 0^+}\frac{-\frac{1}{2}(1+16t^2)^{-1/2}(32\cancel{t})}{2\cancel{t}}$$

$$= \lim_{t\to 0^+}\frac{-8}{\sqrt{1+16t^2}} \overset{\text{direct sub}}{=} \frac{-8}{\sqrt{1+16(0)^2}} = \frac{-8}{1} = -8$$

We will see these in the form of $\lim_{x \to a}\left(f(x)^{g(x)}\right)$.

The Indeterminate Forms 0^0, ∞^0, and 1^∞:

L'Hôpital's Rule *cannot*, once again, be directly used in these cases. We will convert them to the $0/0$ or ∞/∞ form first. These are *not* covered in the homework but here is the procedure and a couple to try for a future where you find yourself facing one down.

Procedure:

1. Consider the limit to be in the form $\lim_{x \to a}\left(f(x)^{g(x)}\right)$. (This limit may also be "as x approaches negative or positive infinity or a from the left or right.)

Analyze $L = \lim_{x \to a}\left(g(x) \cdot \ln\left(f(x)\right)\right)$. This will be in the form $0 \cdot \infty$; convert it to $0/0$ or ∞/∞ and use l'Hôpital's Rule.

2. If L is finite, then $\lim_{x \to a}\left(f(x)^{g(x)}\right) = e^L$.

This e is from natural exponential function fame.

If L is ∞, then $\lim_{x \to a}\left(f(x)^{g(x)}\right) = \infty$.

If L is $-\infty$, then $\lim_{x \to a}\left(f(x)^{g(x)}\right) = 0$.

Note: For 1^∞, L will be in the form $\infty \cdot 0$.

For 0^0, L will be in the form $0 \cdot (-\infty)$.

For ∞^0, L will be in the form $0 \cdot \infty$.

Recall, the graph of $y = \ln x$.

Simplify, simplify, simplify.

not covered

expl 6: Find the following limit.

$$\lim_{x \to 0^+} x^{2x}$$

Optional Exploration:

not covered

expl 7: Find $\lim_{x\to 0}\left(e^{ax}+x\right)^{1/x}$ for some $a \in \mathbb{R}$.

First, verify its form and identify the functions $f(x)$ and $g(x)$. Follow the procedure, converting it to 0/0 form.

Paul Dawkins tells us that $\frac{d}{dx}\left(\ln\left(f(x)\right)\right) = \frac{f'(x)}{f(x)}$.

Find L in terms of a. Is it finite? Then go to step 2 of the procedure to find the limit.

Pitfalls of l'Hôpital's Rule:

1. Be sure you are applying the rule correctly and finding the derivatives of the correct parts.

2. Your first step should always be to verify that you, in fact, have an indeterminate form.

3. As in life, simplify. Be sure to simplify in between iterations of l'Hôpital's Rule. Evaluate the limit as soon as it is no longer an indeterminate form.

4. Sometimes, repeated use of l'Hôpital's Rule will lead to unending cycles. If that happens, stop flogging that dead horse and find another method.

5. If, after using l'Hôpital's Rule, you end up at a limit that does *not* exist, it does *not* necessarily mean that the original one does *not* exist. Consider $\lim_{x\to\infty}\frac{3x+\cos x}{x} = \lim_{x\to\infty}\frac{3-\sin x}{1}$. Do you see what we did from the first to the second limit? What is the value of the second limit? dne
(The first one is actually equal to 3.) took ders of top and bottom