Calculus I
Class notes
The Derivative as a Function (section 3.2)

Deeper into the woods we go …

Do you remember what makes a relationship a function?

Functions are special because we know that an input results in *exactly* one output. There is *no* ambiguity when assigning a y-value for an x-value.

We have found the derivative of a function at a certain x-value. Here, we generalize this idea to find a formula for the derivative of a function for whatever x-value that may come up.

Below we see a function with a few tangent lines drawn. As we have seen, the slopes of these tangent lines (derivatives) depend on the shape of the graph.

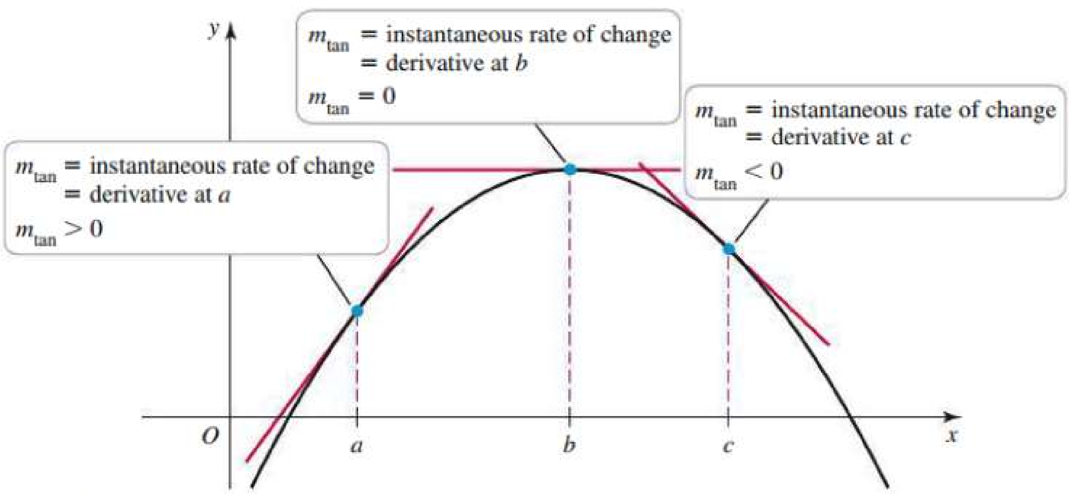


Figure 3.14

On the left, the tangent line slopes upward and so the derivative at $x = a$ would be positive.
On the right, the tangent line slopes downward and so the derivative at $x = c$ would be negative.
Do you see why we say the derivative is 0 in the middle where $x = b$?

Definition: Derivative Function: The **derivative of *f***, denoted $f'(x)$, is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ provided this limit exists. This assumes x is in the domain of f.

The derivative function is "derived" from the original function.

As we have seen, we will use the term **differentiable at *x*** to mean that $f'(x)$ exists. If $f'(x)$ exists for all $x \in I$, we say f is **differentiable on *I***. (Here, I is an interval of x-values.)

If f is a smooth curve, we can find a value for $f'(x)$ for any x in the domain. We will look at this more later.

expl 1: For the following function, use limits to find the derivative function $f'(x)$. Then use it to evaluate $f'(-1)$ and $f'(4)$.

$f(x) = x^2 + 3x$

Later, we learn shortcut formulas. Now, we practice with the limit definition.

expl 2a: For the following function, use limits to find the derivative function $f'(w)$. Then use it to evaluate $f'(1)$ and $f'(3)$.

$f(w) = \sqrt{4w-3}$

It's going to start off nasty. Multiply both top and bottom by the conjugate of the top.

Remember that the slope of the tangent line changes as we progress across the graph. That is why $f'(w)$ is found to be a function. When we substitute specific w-values, we find the slopes of specific tangent lines.

Derivative Notation:

Since h is the amount of change from one x-value to another on a secant line, we could think of this difference as Δx.

This is the Greek letter delta and can be read as "change in".

Using this new symbol, we could write this slope as $m_{\sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$. It's not that big of a leap to use this notation to rewrite

$$f'(x) = \lim_{\Delta x \to 0} m_{\sec} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

Okay, so we're to the important bit. We may use $\frac{dy}{dx}$ to mean $f'(x)$. In certain problems, this is very helpful.

You might also see notation such as

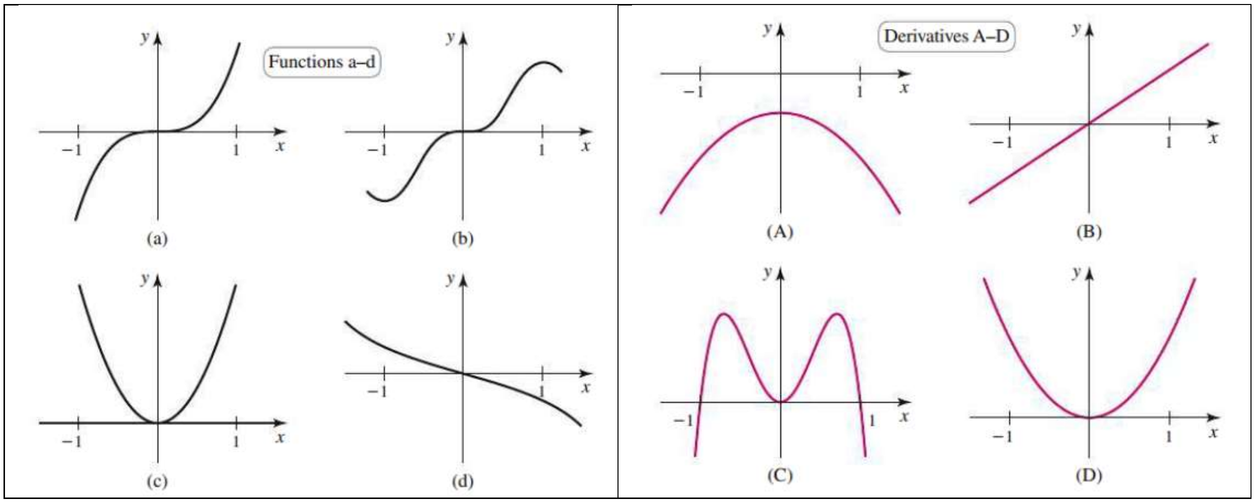
$$\frac{df}{dx}, \quad \frac{d}{dx}\big(f(x)\big), \quad D_x\big(f(x)\big), \quad y'(x), \quad \ldots$$

To show that we are evaluating a derivative at a specific x-value a, we might write $\left.\frac{df}{dx}\right|_{x=a}$ or $y'(a)$, among others. Most common in this class will be good old $f'(a)$.

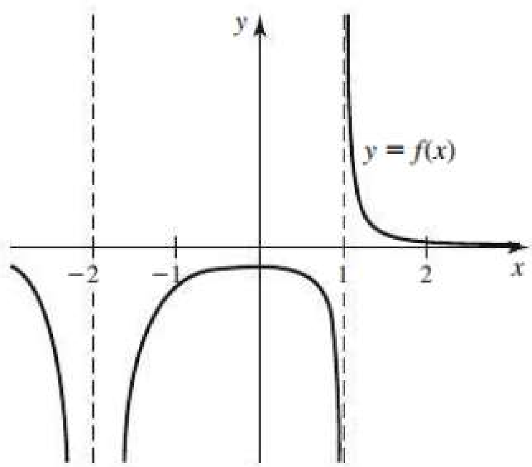
expl 2b: Use some notations from above to rewrite $f'(w)$ and $f'(3)$ from the previous page.

Matching Derivative Graphs to Those of Original Functions:
We must keep in mind that the graph of a function determines if its derivative will be positive or negative or zero (seen on page 1). We will explore more nuance later but that's good for now.

expl 3: The left graphs are functions and the right graphs are their derivatives. Match each graph on the left to one graph on the right.



expl 4: To the right, we see the graph of a function. Draw a possible graph of its derivative on the same axes.
Give yourself some notes to help you remember what you did.



A Logic Sidebar:

Definition: Conditional Statement: An "If, then" statement like many of our theorems have been. Often, we write a generic conditional as "If p, then q."

Definition: Contrapositive Statement: The contrapositive of the above statement is "If *not* q, then *not* p."

If a conditional statement is true, then its contrapositive is always true as well. A standard example of this is the following.

> Assume "If it rains, then I will drive you home." is true.
> Notice that "If I *don't* drive you home, then it did *not* rain." *must* also be true. The reasoning being if it rains, then you were driven home. Since I did *not* drive you home, then it could *not* have rained.

Connection Between Continuity and Differentiability:

We have this theorem and its contrapositive.

Both are true statements.

Theorem 3.1: If f is differentiable at a, then f is continuous at a.

Contrapositive: If f is *not* continuous at a, then f is *not* differentiable at a.

A Common Misconception: The logic flow of a theorem is important. The statement "If f is continuous at a, then f is differentiable at a." is *NOT* true.

Here is an example that shows this to be false.

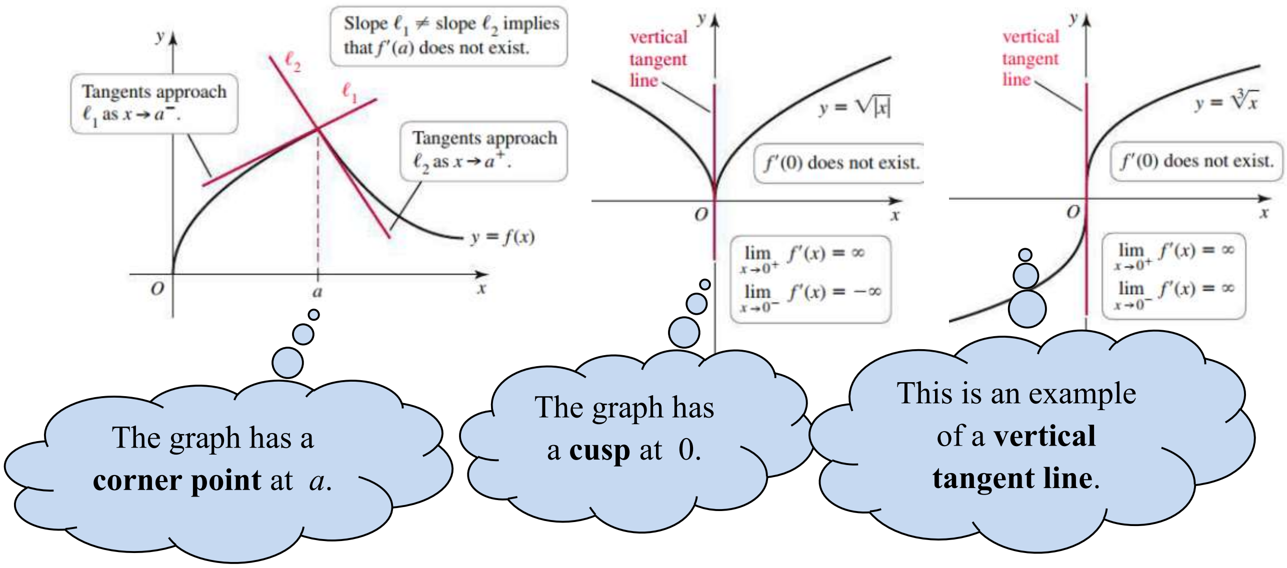
Draw a quick graph of $y = |x|$. Notice it is continuous over its entire domain $(-\infty, \infty)$. But is it differentiable for all x values?

We know $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$ or, rather, consider the point (0, 0) and another point very close by.

Move this point closer and closer to (0, 0) to find this limit from the left? From the right? Do they match? What does that mean about $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$?

So, even though this function is continuous over $(-\infty, \infty)$, it is *not* differentiable at $x = 0$.

More Examples:
Here are three scenarios that can cause a function to *not* be differentiable at a point.



The book provides us with a nice summary which you might find useful.

When Is a Function Not Differentiable at a Point?

A function f is *not* differentiable at a if at least one of the following conditions holds:

a. f is not continuous at a (Figure 3.24).

b. f has a corner at a (Figure 3.25).

c. f has a vertical tangent at a (Figure 3.26).

Referenced figures can be found in the book.

We will wrap with this distinction.

Continuity (for a function *f*) requires $\lim_{x \to a} f(x) = f(a)$ or written in another way,

$\lim_{x \to a} \left(f(x) - f(a) \right) = 0$.

Differentiability requires more. It requires that $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.