

Recall that the derivative is the slope of the tangent line at a certain point. This slope tells us how quickly the function is changing.

We have already seen how if $s(t)$ is the position function of an object in motion, then its derivative $s'(t)$ tells us the object's velocity, or rate of change of its position.

We have seen that the units for this rate of change are based on the fact that this is no more than slope (difference of y -values divided by difference of x -values). Hence if $s(t)$ is in feet and t is in seconds, then $s'(t)$ would be in feet per second.

We will start to use the function name $v(t) = s'(t)$ to mean the object's **velocity**. What is the rate of change of this velocity function? In other words, what would we call $v'(t) = s''(t)$? What would the units be for such a function?

Don't forget about the alternative notation $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$.

We will cover various functions and their rates of change (derivatives). A commonly explored one is that $s(t)$, $v(t)$, and $a(t)$ we started with. We will also play with cost, profit, and revenue and their derivatives **marginal cost, profit, and revenue**. Since we're down that road, we'll also learn about **average cost, profit, and revenue**.

Definition: Displacement: how far an object has travelled. The **displacement** of an object between $t = a$ and $t = a + \Delta t$ is $\Delta s = s(a + \Delta t) - s(a)$. We call this Δt the **elapsed time**.

Definition: Speed: **Speed** is the magnitude of velocity. Algebraically, we say that **speed** = $|v|$.

This is pronounced "delta t".

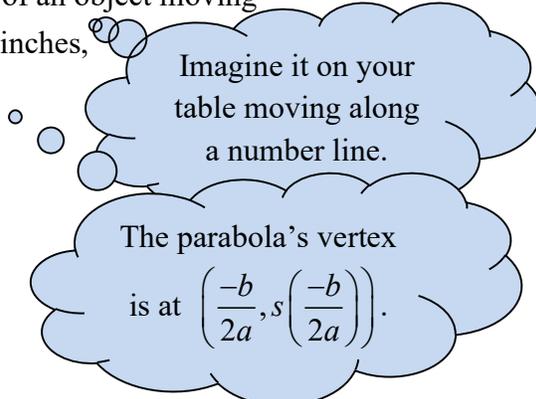
Velocity is a **vector** quantity. It has magnitude *and* direction. Speed is **scalar** and only describes magnitude.

An object's velocity might be -70 ft/sec. The negative sign indicates it's going down. Its speed is said to be 70 ft/sec (removing the direction).

Horizontal Motion:

expl 1: Suppose $s = f(t) = t^2 - 4t$, $0 \leq t \leq 5$, gives the position of an object moving horizontally along a line after t seconds. Here, s is measured in inches, with $s > 0$ corresponding to positions right of the origin.

a.) Graph the position function $f(t)$ for the given interval.



b.) Analyze the graph above to find when the object is stationary, moving to the left, and moving to the right?

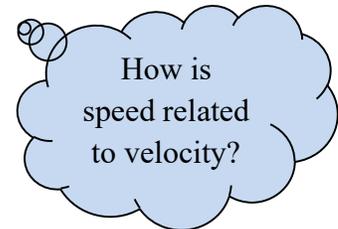
c.) Find and graph the velocity function. Label the parts of this graph that correspond to the object being stationary, moving to the left, and moving to the right.

expl 1 (continued):

d.) Determine the object's velocity and acceleration when $t = 1$ second.

e.) Determine the object's acceleration when its velocity equals 0 inches per second.

f.) Find the object's **speed** for all integer values in the interval $[0, 5]$. Use your table to graph the speed function. On what interval(s) is the object's speed increasing?



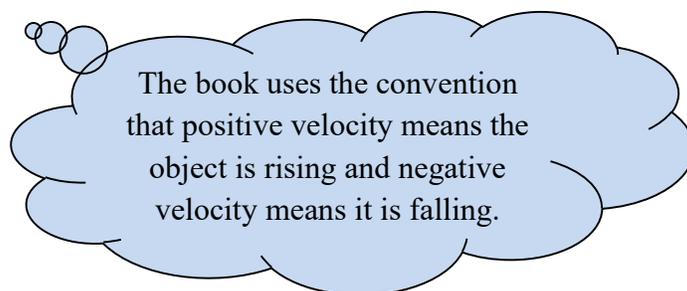
Motion in Gravitational Fields:

expl 2: We throw a stone upward off a cliff edge and let it fall to the beach below. (No people were harmed during the making of this example.) Its position (height, in feet) is given by

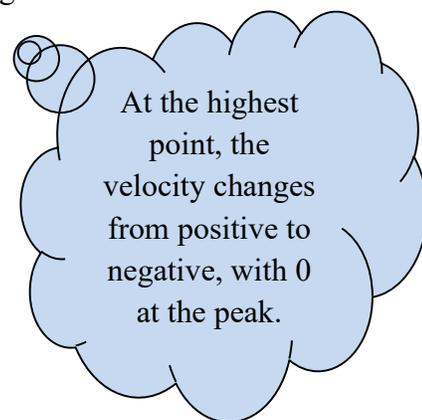
$$s(t) = -16t^2 + 64t + 32 \text{ where } t \text{ is time in seconds.}$$

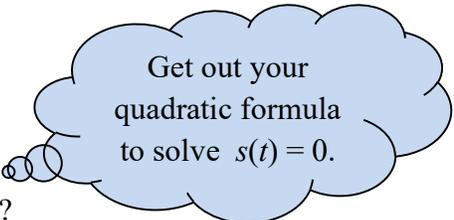
a.) Draw a rough graph of $s(t)$.

b.) Find and graph the stone's velocity function $v(t)$. Fill in details as you discover them.



c.) When does the stone reach its highest point? What is the maximum height it achieves?





Get out your
quadratic formula
to solve $s(t) = 0$.

expl 2 continued:

d.) When does the stone hit the beach below?

e.) With what velocity does the stone hit the beach?

f.) On what intervals is the stone's speed increasing? Give exact values.

Rates of Change, in General:

We are working with specific examples here but understand that these ideas will work for any function. Let's say you know a function $p(t)$ measures something like a population *or* price per item *or* number of internet users, etc. Here, t is taken to be time where $t \geq 0$.

This function's **instantaneous rate of change** is its derivative $p'(t)$ or $\frac{dp}{dt}$. This measures how fast $p(t)$ is changing with respect to time.

The units of this rate of change will always be "units of $p(t)$ divided by units of t ".

Marginal and Average Cost, Profit, and Revenue:

Let $C(x)$ be the cost of producing x items. Perhaps we will be given the function $C(x) = -0.02x^2 + 50x + 100$ dollars.

The **Average Cost per Item** would be given by $\bar{C}(x) = \frac{C(x)}{x}$.

The **Marginal Cost** is the approximate cost to produce *one additional item* after producing x items. And that sounds like rate of change. Hence, marginal cost is $C'(x)$.

Further, let $p(x)$ denote the sale price per item if x items are sold.

We would find **Revenue** (which is total money brought in) as the product $R(x) = x \cdot p(x)$.

Which leaves us **Profit** (the money we really made) as

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= x \cdot p(x) - C(x) \end{aligned}$$

Here's the *uppercase P*.

We are in the US so our units for cost, revenue, and profit will usually be in dollars with the units of their derivatives derived accordingly.

Recall that profit equals revenue minus cost.

Pronounced "C bar".

This should make sense if you picture a total cost of \$500 for 100 items.

Notice this is *lowercase p*.

It is *not* a long stretch to define average or marginal revenue and profit as you see here.

The **Average Revenue per Item** would be given by $\bar{R}(x) = \frac{R(x)}{x}$. The **Marginal Revenue**, $R'(x)$, is the approximate revenue gained by selling *one additional item* after selling x items.

The **Average Profit per Item** would be given by $\bar{P}(x) = \frac{P(x)}{x}$. The **Marginal Profit**, $P'(x)$, is the approximate profit gained by selling *one additional item* after selling x items.

expl 3: A company has a cost function of $C(x) = -0.02x^2 + 50x + 100$ dollars when they produce x items. The sale price per item is given as $p(x) = 100$ dollars.

a.) Find the profit function $P(x)$.

b.) Find the average profit and marginal profit functions.

c.) Find the average profit and marginal profit when $x = 500$ items are sold. Interpret.

