Many functions are the composition of two functions whose derivative rules we know.

Calculus I
Class notes
The Chain Rule (section 3.7)

What's the derivative of the function $y = e^{-3x}$?

We know that $\frac{d}{dx}\left(e^x\right) = e^x$ but we *cannot* simply say that $y' = e^{-3x}$ because the exponent is *not* just $x$. In fact, this exponent of $-3x$ can be thought of as a function unto itself. And, that's the key. We need to start seeing some functions as the composition of two functions whose derivative rules we know.

We can think of $y = e^{-3x}$ as $f(g(x))$ where $g(x) = -3x$ and $f(u) = e^u$. Find the formula for the composed function $f(g(x))$ to verify that.

You can review "function composition" by searching for it on [www.khanacademy.org](http://www.khanacademy.org).

For this and other complicated functions, we will use the Chain Rule. Often, algebra along with our previous rules can be used like in the case of $\frac{d}{dx}\left((2x+6)^3\right)$. How would you do that?

However, once learned, the Chain Rule will prove easier. By the way, do you see the functions $f$ and $g$ that make up the function $h(x) = (2x+6)^3$?

**THEOREM 3.12 The Chain Rule**

Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at $x$. The composite function $y = f(g(x))$ is differentiable at $x$, and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} \qquad (1)$$

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\cdot g'(x) \qquad (2)$$

The derivative of $f$ evaluated at $g$ multiplied by the derivative of $g$ evaluated at $x$.

Once you get the hang of it, this will come naturally. The book has this handy set of steps to help the beginner.

**PROCEDURE Using the Chain Rule**

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify an outer function $f$ and an inner function $g$, and let $u = g(x)$.
2. Replace $g(x)$ with $u$ to express $y$ in terms of $u$:

$$y = f(\underbrace{g(x)}_{u}) \Rightarrow y = f(u).$$

3. Calculate the product $\dfrac{dy}{du} \cdot \dfrac{du}{dx}$.
4. Replace $u$ with $g(x)$ in $\dfrac{dy}{du}$ to obtain $\dfrac{dy}{dx}$.

It does feel cumbersome at first but you will learn to do it with more fluidity.

expl 1a: Find $y'$ for $y = e^{-3x}$.

Do you recall the exponent equivalency for roots?

expl 1b: Find $y'$ for $y = \sqrt{-3x}$.

**Chain Rule for Powers:**

Since we do this a lot, it is somewhat helpful that the book draws out this specifically for its own formula. In fact, it is just the Chain Rule combined with the Power Rule from earlier.

**THEOREM 3.13 Chain Rule for Powers**

If $g$ is differentiable for all $x$ in its domain and $p$ is a real number, then

$$\frac{d}{dx}\left((g(x))^p\right) = p(g(x))^{p-1}g'(x).$$

expl 2: Find $\frac{d}{dx}\left((2x+6)^3\right)$.

expl 3: Find $\frac{d}{dx}\left(\sqrt{x^2+1}\right)$.

expl 4: Find $y'$ for $y = x(x+2)^{1/3}$.

Product Rule, anyone?

expl 5: Find $y'$ for $y = \tan\left(5x^2\right)$.

expl 6: Find $y'$ for $y = \sin^5\left(\cos(3x)\right)$.

This is also written as
$y = \left(\sin\left(\cos(3x)\right)\right)^5$.

*Three* instances
of the Chain
Rule?!?

expl 7: The total energy in megawatt-hours (MWh) used by a town is given by

$E(t) = 400t + \frac{2400}{\pi}\sin\left(\pi t/12\right), \quad t \geq 0$ where $t$ is measured in hours with $t = 0$ corresponding to noon.

a.) Find the power, or the *rate* of energy consumption, $P(t) = E'(t)$, in terms of megawatts.

b.) Graph $P(t) = E'(t)$ in the window [0, 24] x [0, 700].

Notice the slopes of tangent lines at the maxes and min are 0.

c.) Algebraically find the time of day that the power is at a maximum? What is the power at that time?

We need $E''(t)$. Do you see why?

Continued discussion of part *c* is on the next page.

expl 7c continued:

So far, we should be at $\sin\left(\pi t/12\right)=0$. Do you recall the graph of $y=\sin(\theta)$ for $0\le\theta\le 2\pi$?

Draw it here to help find the $t$-values that make $\sin\left(\pi t/12\right)=0$.

d.) Your work also revealed a minimum power. When does that occur and at what time will the city see that? What is the minimum power?