

We need a solid, mathematically defined concept for limit. We may see an even more rigorous definition later.

Calculus I  
Class notes  
Definitions of Limits (section 2.2)

A later section (which we may cover) will add rigor to this concept. For now, this will do. We will define the limit of a function at a certain value of  $x$  (independent variable).

**Definition: Limit of a Function:**

Suppose a function  $f$  is defined for all values of  $x$  near  $x = a$  except possibly at  $a$ .

That sounds a little crazy but what we want to picture is a function like this one here. Notice it is defined (shown by the points of the graph) for all values except  $x = 3$ . So, think of  $a$  as any input, even 3.

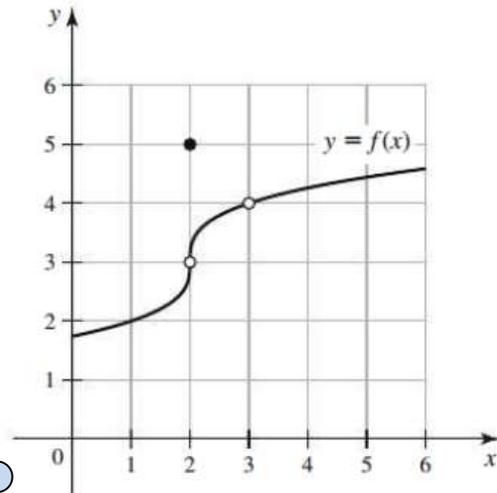


Figure 2.7

If  $f(x)$  is arbitrarily close to  $L$  (meaning as close to  $L$  as we would like) for all  $x$  sufficiently close (but *not* equal) to  $a$ , we write  $\lim_{x \rightarrow a} f(x) = L$ .

This will make more sense as you practice.

Pronounced “the limit, as  $x$  approaches  $a$ , of  $f(x)$ ”.

expl 1: Consider the graph given above. Find the following.

a.)  $f(1)$  and  $\lim_{x \rightarrow 1} f(x)$

b.)  $f(2)$  and  $\lim_{x \rightarrow 2} f(x)$

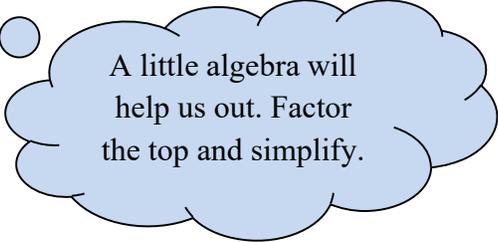
c.)  $f(3)$  and  $\lim_{x \rightarrow 3} f(x)$

For a limit to exist, the function should approach a single  $f(x)$  value from both the left and right sides. We formalize this later.

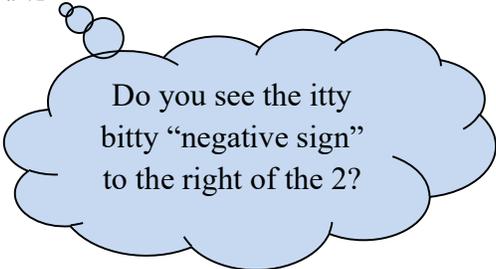
**Finding Limits from a Table of Values:**

expl 2: Calculate the following values for  $f(x) = \frac{x^2 - 4}{x - 2}$  and use it to conjecture about the value of  $\lim_{x \rightarrow 2} f(x)$ .

$x$	1.9	1.99	1.999	1.9999
$f(x)$				

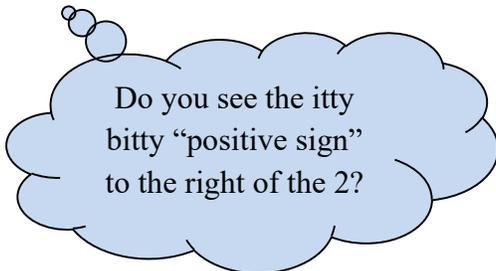


As  $x$  approaches 2 from the left, what does the table indicate  $f(x)$  is approaching? We will write this as  $\lim_{x \rightarrow 2^-} f(x)$ . Practice the notation here.



$x$	2.1	2.01	2.001	2.0001
$f(x)$				

As  $x$  approaches 2 from the right, what does the table indicate  $f(x)$  is approaching? We will write this as  $\lim_{x \rightarrow 2^+} f(x)$ . Practice the notation here.



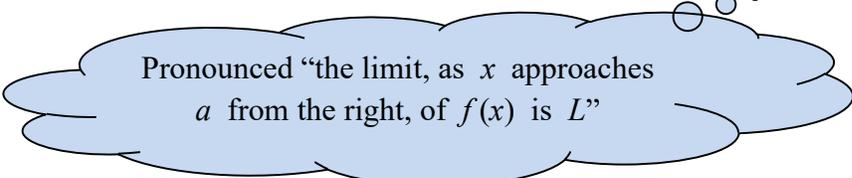
As we will see in the next theorem, if these two (one-sided) limits agree, then the limit of  $f(x)$  as  $x$  approaches 2 is that same number. Practice the notation by writing that now.

Let's fill in some details with formal definitions.

## One-sided Limits:

### Definition: Right-sided Limit:

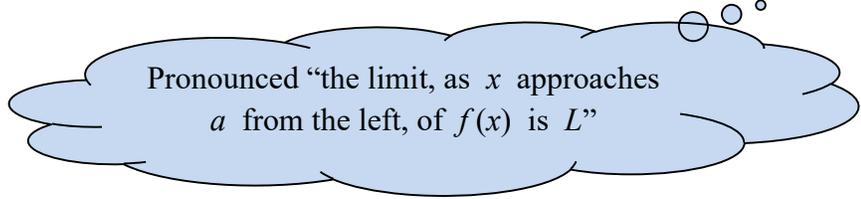
Suppose a function  $f$  is defined for all values of  $x$  near  $x = a$  with  $x > a$ . If  $f(x)$  is arbitrarily close to  $L$  for all  $x$  sufficiently close to  $a$  with  $x > a$ , we write  $\lim_{x \rightarrow a^+} f(x) = L$ .



Pronounced “the limit, as  $x$  approaches  $a$  from the right, of  $f(x)$  is  $L$ ”

### Definition: Left-sided Limit:

Suppose a function  $f$  is defined for all values of  $x$  near  $x = a$  with  $x < a$ . If  $f(x)$  is arbitrarily close to  $L$  for all  $x$  sufficiently close to  $a$  with  $x < a$ , we write  $\lim_{x \rightarrow a^-} f(x) = L$ .

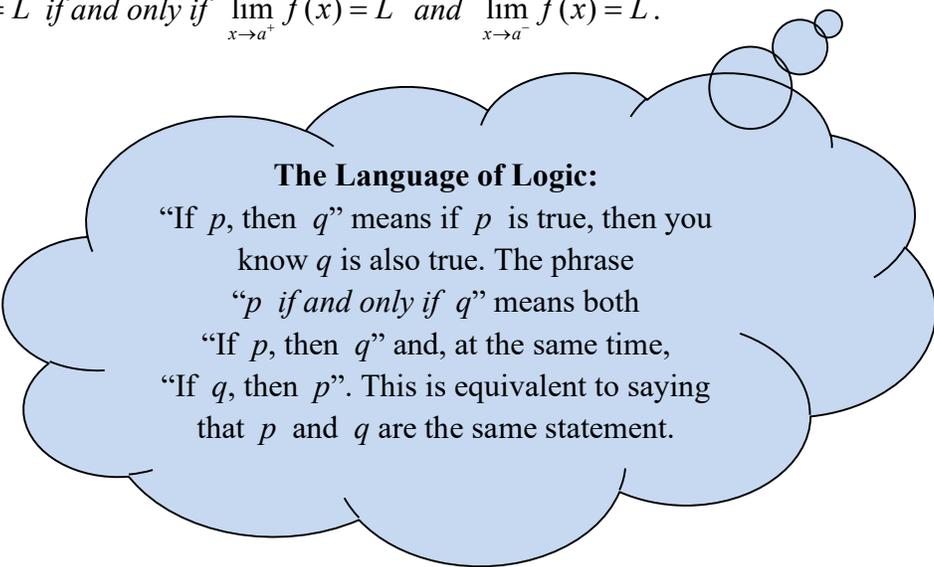


Pronounced “the limit, as  $x$  approaches  $a$  from the left, of  $f(x)$  is  $L$ ”

As you might have guessed from the last example, we can deduce the limit of  $f(x)$  as  $x$  approaches  $a$  if we know both of these one-sided limits. Here’s a theorem.

### Theorem 2.1: Relationship between one-sided and two-sided limits:

Suppose a function  $f$  is defined for all values of  $x$  near  $x = a$  except possibly at  $a$ . Then  $\lim_{x \rightarrow a} f(x) = L$  if and only if  $\lim_{x \rightarrow a^+} f(x) = L$  and  $\lim_{x \rightarrow a^-} f(x) = L$ .



#### The Language of Logic:

“If  $p$ , then  $q$ ” means if  $p$  is true, then you know  $q$  is also true. The phrase “ $p$  if and only if  $q$ ” means both “If  $p$ , then  $q$ ” and, at the same time, “If  $q$ , then  $p$ ”. This is equivalent to saying that  $p$  and  $q$  are the same statement.

So, to find a limit such as  $\lim_{x \rightarrow 2} f(x)$ , we must find *both*  $\lim_{x \rightarrow 2^+} f(x)$  and  $\lim_{x \rightarrow 2^-} f(x)$ . If they agree, then the limit “as  $x$  approaches 2” exists and is this common number.

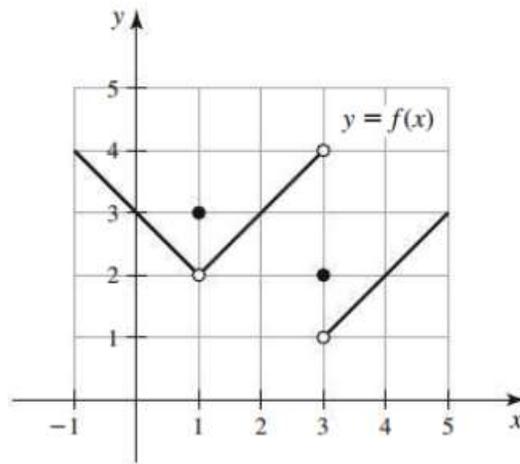
expl 3: Find the following using the graph to the right.

a.)  $f(3)$

b.)  $\lim_{x \rightarrow 3^-} f(x)$

c.)  $\lim_{x \rightarrow 3^+} f(x)$

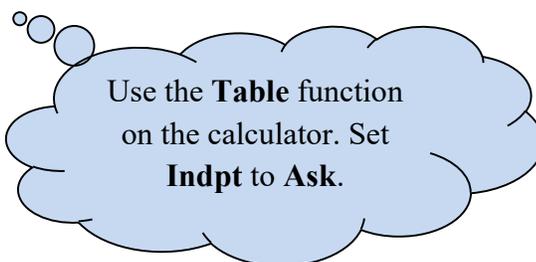
d.)  $\lim_{x \rightarrow 3} f(x)$



Graphs and tables can be deceiving or hard to interpret. Be you careful!

expl 4: Create a table of values for  $y = \tan\left(\frac{3}{x}\right)$  as started below. Describe the pattern. What happens to  $y = \tan\left(\frac{3}{x}\right)$  as  $x$  approaches 0? In other words, what is  $\lim_{x \rightarrow 0} \tan\left(\frac{3}{x}\right)$ ?

$x$	$y = \tan\left(\frac{3}{x}\right)$
$\frac{12}{\pi}$	
$\frac{12}{3\pi}$	
$\frac{12}{5\pi}$	
$\frac{12}{7\pi}$	
$\frac{12}{9\pi}$	



What the heck is going on here? It does *not* look as though the function is approaching any one value as  $x$  approaches 0?!? Does graphing the function help?

Try the **Standard** window. Then **Zoom In** more and more. Does that help?

Eek! It turns out that this limit does *not* exist. It is hard to see that from the graph. We will need other analytical methods to decipher some limits. More on that later.